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APM 504 Project: Random Walks as Electrical Networks
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Background

Walks on Electrical Networks

To study random walks on graphs, we can realize these graphs as electrical networks and use properties of these networks to derive properties of the graph. To turn a graph into an electrical network, we must make connections between aspects of a graph and those of an electrical network. Let $G = (V, E)$ be a connected graph and $x, y \in V$. First, we assign a conductance to each edge, meaning that we have a function on the graph, $c : E \to [0, \infty]$, where the conductance along an edge is denoted by $c(x, y)$. We also define $c(x) = \sum_{z\sim x} c(x, z)$, where $z \sim x$ means that x and z are connected by an edge. This then gives us a random walk on the electrical network (G, c). In this random walk, the transition probability between x and y is given by

$$p(x, y) = \frac{c(x, y)}{c(x)} \mathbf{1}(x, y) \in E.$$

This transition probability makes this random walk a discrete-time Markov chain.

To further explore the connection between random walks and electrical networks, we can include more aspects of electrical networks into our graph. First, we can choose two vertices, a and b on the graph and apply a voltage of 1 at a and 0 at b (by putting a one-volt battery between them). Then, we can define the voltage at any vertex, x, as $v(x)$, and the current along the edge connecting x and y as $i(x, y)$. These are connected by Ohm's Law, which gives us that for any $(x, y) \in E$,

$$i(x, y) = \frac{v(x) - v(y)}{R(x, y)} = c(x, y)(v(x) - v(y)).$$

It can also be shown that the voltage, $v(x)$ is a harmonic function. So, when a voltage of 1 is applied to a and a voltage of 0 applied to b, the voltage at any vertex, x, can be interpreted as the probability that a random walk starting from x reaches a before b. So, for $x \in V$,

$$v(x) = P_x(\tau_a < \tau_b), \quad \text{where } \tau_y = \inf\{n \geq 0 : X_n = y\},$$

with v being a unique harmonic function on the network satisfying this condition. Since v is harmonic, we have that for all $x \neq a, b$,

$$v(x) = \sum_{z\sim x} \frac{c(x, z)}{c(x)} v(z),$$

and $v(a) = 1$, $v(b) = 0$.

For the current, Kirchhoff's current law gives us that the current flowing through any vertex, x, denoted by

$$i(x) = \sum_{z \sim x} i(x, z),$$

such that x is not a or b, is 0. On any edge (x, y), the current has a probabilistic interpretation: for a random walk starting at a, the current, $i(x, y)$, is the expected value of the difference between the number of times a random walk goes from x to y and the times it goes from y to x, before reaching b, divided by the effective resistance.

From the proof for the probabilistic interpretation of current, we can define the effective resistance between a and b as

$$R_{eff}(a, b) = \frac{1}{i(a)} = \frac{1}{\sum_{z \sim a} i(a, z)}.$$

The interpretation of the effective resistance is that it is the voltage needed to send a unit of current from a to b. It is also related to the escape probability, $p_{esc}(a, b)$, which is the probability that a random walk starting at a reaches b before returning to a. This relationship is given by

$$c_{eff} = 1/R_{eff} = c(a)p_{esc}(a, b).$$

Thus, the effective resistance can be used to find the escape probability of the random walk.

Calculating Escape Probabilities

There are two ways to utilize this representation of a random walk as an electrical network connection to find the escape probability of a connected graph. One option is to calculate the voltage at each vertex in the network by solving a linear system for the unique harmonic function $v(x)$. Then, using Ohm's Law, this voltage function can be used to find the total current, $i(a)$, and the effective resistance, which gives the escape probability.

The alternative method involves reducing the network to a single edge connecting a and b. This is done using two rules for resistance which come from electrical networks. The first rule is that two serial edges with resistances R_1 and R_2 can be replaced with a single edge with a resistance equal to the sum of the resistances, $R_1 + R_2$. The second rule states that two parallel edges between the same vertices can be replaced with a single edge that has a conductance equal to the sum of the conductances. In other words, two parallel edges with resistances R_1 and R_2 can be replaced with a single edge with resistance $(1/R_1 + 1/R_2)^{-1}$. To use this method, we first want to reduce a network by identifying edges that are the same distances from both a and b, and model the same number of edges on these fewer vertices where some are identified together. Then, the edges can be reduced and simplified to a single edge using the rules. The resistance of the remaining edge is then the effective resistance between a and b, which can be used to find the escape probability.

The connection here between escape probability and effective resistance can be expanded to properties of infinite graphs with Rayleigh's Monotonicity Law.

Connection to Infinite Networks

To define recurrence and transience in an infinite electrical network, we will use a sphere and the return time to the sphere. So, for every $r > 0$, we define a sphere with center a and radius r by

$$S(a, r) = \{z \in V : d(a, z) = r\},$$

where d is the graph distance function. Then, the first return time to this sphere is given by

$$T(a, r) = inf\{n > 0 : X_n \in S(a, r)\}.$$

Using the return time, define the probability of escape, $p_{esc}(r)$ as

$$p_{esc}(r) = P(T(a, r) < T_a | X_0 = a).$$

Then, since this escape probability is nonincreasing in r, we have that the limit $lim_{r\to\infty} p_{esc}(r)$ exists. So, by the definition of recurrence, an electrical network is recurrent if and only if this limit is zero. This would imply that the walk returns with probability zero.

We can combine this definition of recurrence with Reyleigh's monotonicity law to analyze infinite walks. Reyleigh's monotonicity law states that increasing the resistance of an edge can only increase the resistance between any two vertices in the network while decreasing the resistance of an edge can only decrease resistance between vertices. This means that for a given recurrent network, if we only increase the resistance of the edges, the network will still be recurrent. Similarly if a network is transient, then decreasing the resistances of some of the edges will still produce a transient network.

To help study graphs using Reyleigh's monotonicity law, we can introduce a few terms for graph modifications. First, we can *cut* an edge by setting its resistance to infinity or equivalently, its conductance equal to 0. This is the same as removing the edge and only increases effective resistance. Another action is *shorting* an edge, where the edge is given zero resistance. This has the effect of identifying the two vertices together that were connected by that edge, which can only decrease effective resistance. One other operation we can perform is *adding* an edge to the graph between two vertices.

Combining these actions with Reyleigh's monotonicity law, we can derive that any subgraph of a recurrent graph is still recurrent while any supergraph of a transient one is still transient. This result is easier to see and understand through the connection with electrical networks and is one benefit of this connection between random walks on graphs and electrical networks.

Problems

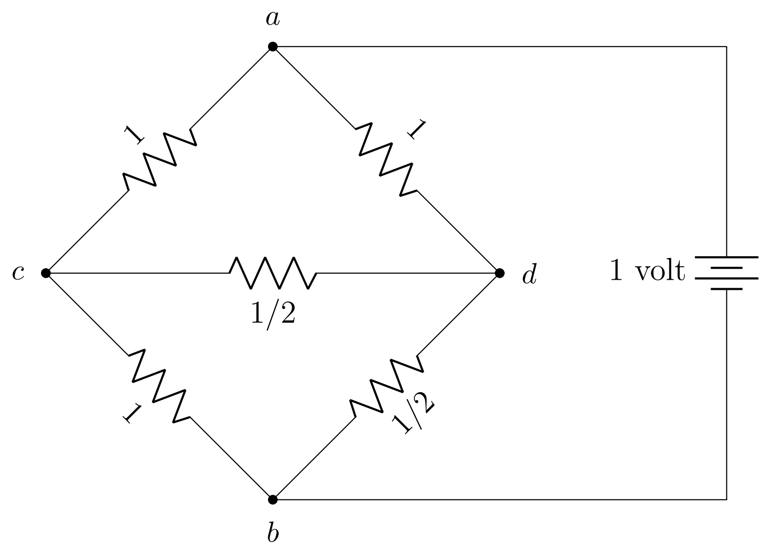


Figure 1 Electrical network for Exercise 8.6.

Exercise 8.6 Consider the electrical network with resistances shown in Figure 1, and put a battery that establishes a voltage one at a and zero at b.

1. Find the voltages at vertices c and d.

From the proof of Lemma 8.3, we know that

$$v(c) = \sum_{x \sim c} \frac{c(c,x)}{c(c)} v(x).$$

Also, we have that the conductance is equal to the recipricoal of the resistance, so $R = 1/c$. So, in this problem we have that

$$c(c) = \sum_{x \sim c} c(c,x) = c(c,a) + c(c,b) + c(c,d) = 1 + 1 + 2 = 4$$

and

$$c(d) = \sum_{x \sim d} c(d,x) = c(d,a) + c(d,b) + c(d,c) = 1 + 2 + 2 = 5.$$

Thus, we can calculate that

$$\begin{aligned}
v(c) &= \sum_{x\sim c} \frac{c(c,x)}{c(c)} v(x) \\
&= \frac{c(c,a)}{4} v(a) + \frac{c(c,b)}{4} v(b) + \frac{c(c,d)}{4} v(d) \\
&= \frac{1}{4}(1) + \frac{1}{4}(0) + \frac{2}{4} v(d) \\
&= \frac{1}{4} + \frac{1}{2} v(d).
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
v(d) &= \sum_{x\sim d} \frac{c(d,x)}{c(d)} v(x) \\
&= \frac{c(d,a)}{5} v(a) + \frac{c(d,b)}{5} v(b) + \frac{c(d,c)}{5} v(c) \\
&= \frac{1}{5}(1) + \frac{2}{5}(0) + \frac{2}{5} v(c) \\
&= \frac{1}{5} + \frac{2}{5} v(c).
\end{aligned}$$

Solving for $v(c)$, we get that $\frac{4}{5}v(c) = \frac{7}{20}$, so $v(c) = \frac{7}{16}$. This gives us that $v(d) = \frac{3}{8}$. Thus, $v(c) = \frac{7}{16}$ and $v(d) = \frac{3}{8}$.

2. Deduce the probability that the random walk on this electrical network reaches vertex b before returning to a when starting from vertex a.

By definition, the probability that the random walk reaches b before returning to a is the escape probability. From the proof of Lemma 8.5, we have that $p_{esc}(a, b) = 1 - \sum_{z\sim a} p(a, z)p(z)$. Here, $p(a, z)$ is the transition probability while $p(z) = v(z)$ by Lemma 8.3. Thus, we can calculate this directly:

$$\begin{aligned}
p_{esc}(a,b) &= 1 - \sum_{z\sim a} p(a,z)p(z) \\
&= 1 - \frac{c(a,c)}{c(a)} p(c) - \frac{c(a,d)}{c(a)} p(d) \\
&= 1 - (\frac{1}{2})(\frac{7}{16}) - (\frac{1}{2})(\frac{3}{8}) \\
&= 1 - \frac{7}{32} - \frac{6}{32} = \frac{19}{32}
\end{aligned}$$

Thus, $p_{esc}(a, b) = \frac{19}{32}$, so the probability that the random walk reaches vertex b before returning to vertex a is $\frac{19}{32}$.

Exercise 8.7 Consider the electrical network obtained from the cube by assigning resistance one to each of the twelve edges. Let a and b be two vertices at graph distance two from each other and let h be the unique harmonic function taking the value one at a and zero at b.

1. Use a probabilistic argument to find (without calculation) the value that the function h takes at each of the vertices x such that $d(a,x) = d(b,x)$.

The probabilistic interpretation of voltage, which is the unique harmonic function in this case, is that the voltage at each vertex is equal to the probability that a random walk starting from this vertex reaches a before b. Now, at the vertices such that $d(a,x) = d(b,x)$, the random walk starting from x is the same number of edges away from a as from b. Since the resistance along each edge of the cube is equal, this random walk is symmetric and has an equal probability to move toward a or toward b at each step. So, the probability that this random walk reaches a first should be equal to the probability that it reaches b first, 1/2. So, h should have a value of $\frac{1}{2}$ at each of the vertices x where $d(a,x) = d(b,x)$.

2. Find more generally the value of h at all vertices.

First, we will categorize the vertices of this graph, identifying vertices that are the same distances away from a and b together since they will have the same value in the function h. Let x_i be the two vertices where $d(a,x_i) = d(b,x_i) = 1$, y_i be the two vertices where $d(a,y_i) = d(b,y_i) = 2$, z be the vertex where $d(a,z) = 1$ and $d(b,z) = 3$, and w be the vertex where $d(a,w) = 3$ and $d(b,x) = 1$. Then, since voltage is harmonic, we can calculate the voltage at each vertex to obtain the value of h at each vertex. We will use the equation derived in the proof of Lemma 8.3:

$$v(c) = \sum_{d \sim c} \frac{c(c,d)}{c(c)} v(d).$$

Note that in this case, since the resistance on each edge is one, the conductance on each edge is also one and so for every vertex c, $c(c) = 1 + 1 + 1 = 3$.

So, carrying out this computation for each (unique) vertex, we obtain the following:

$$v(x_i) = \frac{1}{3}(1) + \frac{1}{3}v(y_i) + \frac{1}{3}(0)$$
$$v(y_i) = \frac{1}{3}v(x_i) + \frac{1}{3}v(z) + \frac{1}{3}v(w)$$
$$v(z) = \frac{1}{3}(1) + \frac{2}{3}v(y_i)$$
$$v(w) = \frac{2}{3}v(y_i) + \frac{1}{3}(0)$$

Since the equations for $v(x_i)$, $v(z)$ and $v(w)$ only depend on $v(y_i)$, we can substitute them into the equation for $v(y_i)$ and solve. Doing so gives us that $v(y_i) = \frac{2}{9} + \frac{5}{9}v(y_i)$, so $v(y_i) = \frac{1}{2}$.

This can then be used to solve for the voltage at each of the other vertices, giving us that $v(x_i) = \frac{1}{2}$, $v(z) = \frac{2}{3}$, and $v(w) = \frac{1}{3}$.

Thus, the unique harmonic function h satisfies that $h(x_i) = h(y_i) = \frac{1}{2}$ for $i = 1, 2$, $h(z) = \frac{2}{3}$, and $h(w) = \frac{1}{3}$.

3. Deduce that the probability that the symmetric random walk on the cube starting at vertex a reaches b before returning to a is equal to 4/9.

By definition, the probability that the random walk reaches b before returning to a is the escape probability. From the proof of Lemma 8.5, we have that $p_{esc}(a,b) = 1 - \sum_{z\sim a} p(a,z)p(z)$. Noting that from Lemma 8.3, $p(z) = v(z)$, with v being the unique harmonic function, so $v = h$ and we can use the harmonic function found in the last part to compute the escape probability.

Now, vertex a is connected to three other vertices in this graph. Using the labels from the last part, these vertices are labeled as x_1, x_2, and z. So, the escape probability is

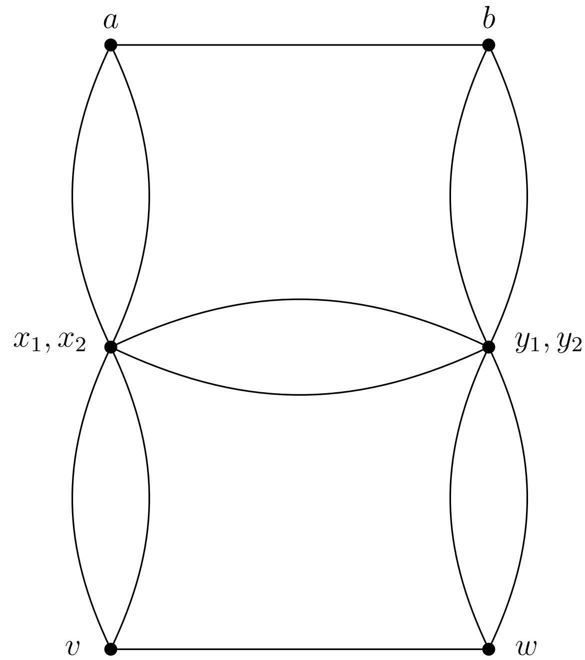
$$\begin{aligned}
p_{esc}(a,b) &= 1 - \sum_{c\sim a} p(a,c)p(c) \\
&= 1 - \sum_{c\sim a} \frac{c(a,c)}{c(a)} p(c) \\
&= 1 - \left(\frac{c(a,x_1)}{c(a)} v(x_1) + \frac{c(a,x_2)}{c(a)} v(x_2) + \frac{c(a,z)}{c(a)} v(z)\right) \\
&= 1 - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \\
&= 1 - \frac{1}{3} - \frac{2}{9} = 1 - \frac{5}{9} = \frac{4}{9}.
\end{aligned}$$

Thus, the escape probability is 4/9, so the probability that the symmetric random walk on the cube starting at vertex a reaches b before returning to a is 4/9.

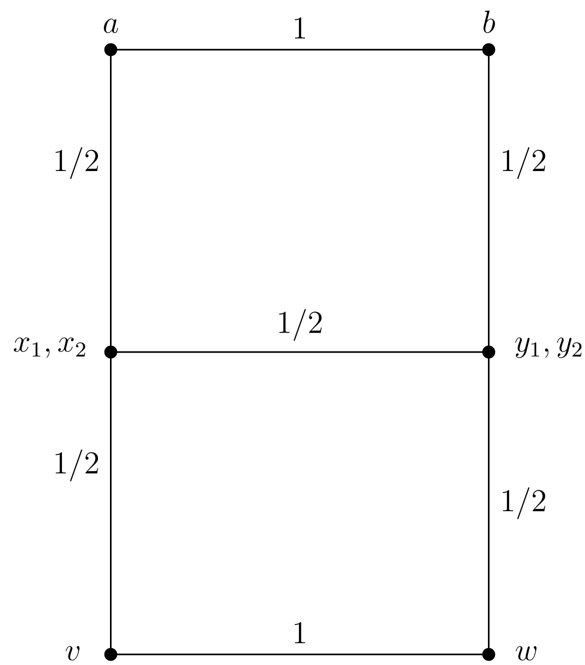
Exercise 8.8 Let a and b be two adjacent vertices of the electrical network obtained from the cube by assigning resistance one to each of the twelve edges.

1. Use the two rules shown in Figure 8.4 as in Example 8.1 to find the effective resistance between a and b.

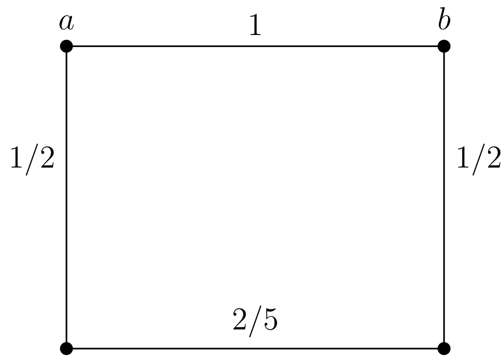
To use the rules in Figure 8.4, we need to reduce the network. Using symmetry, we can identify the two vertices at distance one of a and distance two of b, call these vertices x_i for $i = 1, 2$. Similarly, we can identify the two vertices at distance one of b and distance two of a, labeling them y_i for $i = 1, 2$. Label the last two vertices v and w. Doing this allows us to reduce the the network to one with six vertices and twelve edges as below, with each edge having resistance 1.



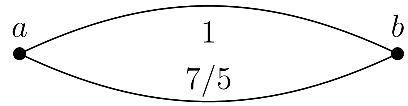
Then, using the resistance rule to reduce parallel edges to a single edge, we can reduce the network to only 7 edges:



Now, we can combine the three serial paths connecting x_1, x_2 to y_1, y_2 via v and w into a single path by summing the resistances as $1/2 + 1 + 1/2 = 2$. Then, we can reduce the two paths connecting x_1, x_2 to y_1, y_2 using the parallel paths rule. So, the resistance of this new path is $(1/2+2)^{-1} = 2/5$. The new reduced network is then



Lastly, we combine the three serial paths connecting a and b into one with a resistance of $1/2+2/5+1/2 = 7/5$, and then reduce the resulting parallel paths using the other resistance rule. This gives a single path between a and b with a resistance of $(1+5/7)^{-1} = 7/12$.



So, we get the the effective resistance between a and b is $R_{eff}(a,b) = \frac{7}{12}$.

2. Deduce the probability that the symmetric random walk on the cube starting at vertex a reaches vertex b before returning to b.

By definition, the probability that the random walk reaches b before returning to a is the escape probability. From Lemma 8.5, we have that

$$1/R_{eff}(a,b) = c_{eff}(a,b) = c(a)p_{esc}(a,b),$$

so $p_{esc}(a,b) = \frac{1}{R_{eff}(a,b)\, c(a)}$. For this network, $c(a) = \sum_{z\sim a} c(a,z) = 1+1+1 = 3$. So, we have that

$$p_{esc} = \frac{1}{R_{eff}(a,b)\, c(a)} = \frac{1}{(7/12)(3)} = \frac{12}{21} = \frac{4}{7}.$$

Thus, $p_{esc}(a,b) = 4/7$, so the probability that the symmetric random walk on the cube starting at vertex a reaches vertex b before returning to b is $4/7$.

3. Compare this escape probability with the two analogous escape probabilities obtained in Example 8.1 and Exercise 8.7.

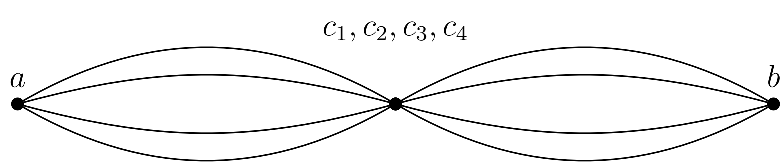
The escape probabilities in each of these exercises use an electrical network on the cube with resistance one on each edge. In Example 8.1, a and b are diagonally opposite corners of the cube, while in Exercise 8.7, a and b are a graph distance of 2 apart. The escape probability for Example 8.1 was 2/5 and for Exercise 8.7, it was 4/9. So, the escape probability for this network is larger than the analogous escape probabilities in those other cases, meaning that a symmetric random walk on the cube is more likely to reach b before returning to a when a and b are closer to each other. The escape probability is higher when the distance between a and b is smaller.

Exercise 8.9 For the symmetric random walks on the octahedron, the icosahedron and the dodecahedron shown in Figure 7.1, compute the probability that the process starting at vertex a reaches vertex b before returning to a where vertices a and b are two diametrically opposite vertices.

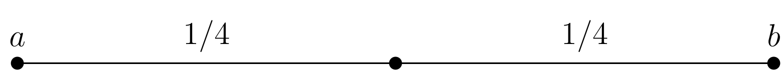
To apply the rules to compute the effective resistance between a and b, I will first categorize the vertices in each of the solids based on their distances from a and b. Then, I can form a network for the system with fewer vertices, which is easier to reduce.

Octahedron

For the octahedron, there are six vertices. We have a and b at opposite vertices and then four vertices that are a distance one from a and a distance one from b. Labeling these vertices c_i and removing the self-loops on the c_i (which are unimportant for this escape probability), we can reduce the graph to the following graph where each edge has resistance 1:



Then, we can use the resistance rules to combine each set of parallel edges into a single edge. In this case, both resultant edges have resistance 4, producing the following reduced graph:



This can be further simplified by combining the paths into a single one connecting a and b, with a resistance of $1/4 + 1/4 = 1/2$. Thus, there is an effective resistance of 1/2 between

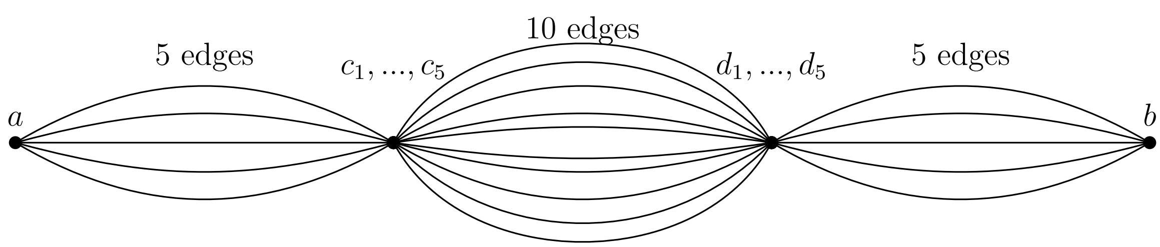
a and b, $R_{eff}(a,b) = 1/2$. We can also compute that $c(a) = \sum_{x\sim a} c(a,x) = 4(1) = 4$. Thus,

$$p_{esc}(a,b) = \frac{1}{R_{eff}(a,b)\, c(a)} = \frac{1}{(1/2)4} = \frac{1}{2}.$$

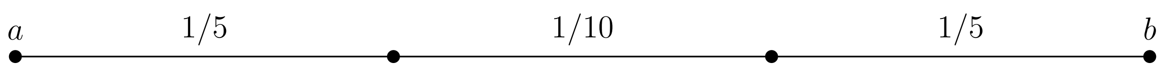
So, for the octahedron, $p_{esc}(a,b) = \frac{1}{2}$.

Icosahedron

The icosahedron has 12 vertices and 30 edges. With a and b at opposite vertices, we have 5 vertices at a distance one from a and a distance two from b, which we will call c_i. There are also 5 vertices a distance two from a and one from b which we will call d_i. With this identification between edges, we can simplify this network by removing the edges between the identified vertices so that we have a graph with 20 edges and four vertices. This graph is displayed below, with each edge having resistance 1.



Using the resistance rules to combine parallel paths, we can reduce the parallel paths to single paths connecting the four vertices. The new resistances are then 1/5, 1/10, and 1/5. The simplified graph is then



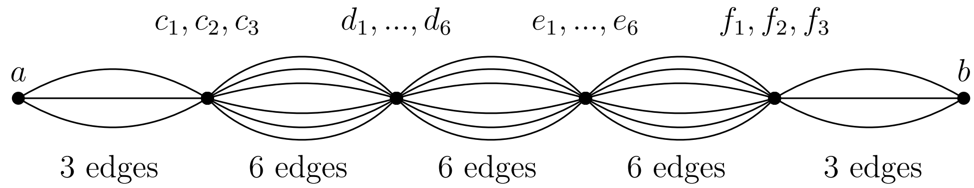
Then, we can use the resistance rules to combine these serial paths between a and b to obtain a single path with a resistance of $1/5 + 1/10 + 1/5 = 1/2$. Thus, $R_{eff}(a,b) = 1/2$. We can also compute that $c(a) = \sum_{x\sim a} c(a,x) = 5(1) = 5$. Thus,

$$p_{esc}(a,b) = \frac{1}{R_{eff}(a,b)\, c(a)} = \frac{1}{(1/2)5} = \frac{2}{5}.$$

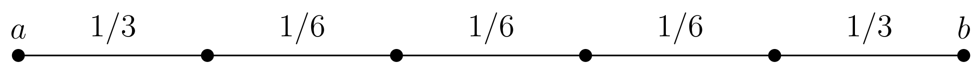
So, for the icosahedron, $p_{esc}(a,b) = \frac{2}{5}$.

Dodecahedron

The dodecahedron has 20 vertices and 30 edges. With a and b at opposite vertices, we can identify the remaining 18 vertices into four categories. There are 3 at a distance 1 from a and 4 from b, denoted c_i, 6 vertices at a distance 2 from a and 3 from b denoted d_i, 6 vertices at a distance 3 from a and 2 from b denoted e_i, and 3 at a distance 4 from a and 1 from b denoted f_i. With this identification, this network can be simplified to a graph with six vertices and 30 edges between them, each with resistance 1.



Combining parallel paths according to the resistance rule, we get the following network with the labeled resistances:



Lastly, combining this path into a single path using the resistance rules, we get an effective resistance of $2(1/3) + 3(1/6) = 7/6$ between a and b. Thus, $R_{eff}(a, b) = 7/6$. We can also compute that $c(a) = \sum_{x \sim a} c(a, x) = 3(1) = 3$. Thus,

$$p_{esc}(a, b) = \frac{1}{R_{eff}(a, b)\, c(a)} = \frac{6}{7}(\frac{1}{3}) = \frac{2}{7}.$$

So, for the dodecahedron, $p_{esc}(a, b) = \frac{2}{7}$.