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APM 502 (Differential Equations II) Project
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**2.7** Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever $u$ is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

**Proof.**

Poisson's formula for the ball is

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y), \;\; x \in B^0(0,r).$$

Using the average value over the integral, this formula can be rewritten as

$$u(x) = r^{n-2}(r^2-|x|^2) \quad {}_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y).$$

Then, we can compute that

$$u(0) = r^n \quad {}_{\partial B(0,r)}\frac{g(y)}{|y|^n}dS(y) = \quad {}_{\partial B(0,r)} g(y)dS(y)$$

since $|y| = r$ on $\partial B(0,r)$. Additionally, we note that for $y \in \partial B(0,r)$, $|x-y| \geq ||x|-|y|| = ||x|-r| = r-|x|$ and $|x-y| \leq |x|+|y| = r+|x|$. Thus,

$$ {}_{\partial B(0,r)}\frac{g(y)}{(r+|x|)^n}dS(y) \leq \quad {}_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \leq \quad {}_{\partial B(0,r)}\frac{g(y)}{(r-|x|)^n}dS(y).$$

Combining these facts, we have that

$$\begin{aligned} r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) &= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}} \quad {}_{\partial B(0,r)} g(y)dS(y) \\ &= r^{n-2}\frac{r^2-|x|^2}{(r+|x|)^n} \quad {}_{\partial B(0,r)} g(y)dS(y) \\ &= r^{n-2}(r^2-|x|^2) \quad {}_{\partial B(0,r)}\frac{g(y)}{(r+|x|)^n}dS(y)dS(y) \\ &\leq r^{n-2}(r^2-|x|^2) \quad {}_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \\ &= u(x). \end{aligned}$$

Similarly,

$$
\begin{aligned}
r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0) &= r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}} {}_{\partial B(0,r)} g(y)dS(y) \\
&= r^{n-2}\frac{r^2-|x|^2}{(r-|x|)^n} {}_{\partial B(0,r)} g(y)dS(y) \\
&= r^{n-2}(r^2-|x|^2) {}_{\partial B(0,r)} \frac{g(y)}{(r-|x|)^n}dS(y)dS(y) \\
&\geq r^{n-2}(r^2-|x|^2) {}_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}dS(y) \\
&= u(x).
\end{aligned}
$$

Thus, we have shown that

$$
r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0).
$$

**2.23** Let $S$ denote the square lying in $\mathbb{R}$ x $(0, \infty)$ with corners at the points $(0,1)$, $(1,2)$, $(0,3)$, $(-1,2)$. Define

$$f(x,t) := \begin{cases} -1 & \text{for } (x,t) \in S \cap \{t > x+2\} \\ 1 & \text{for } (x,t) \in S \cap \{t < x+2\} \\ 0 & \text{otherwise.} \end{cases}$$

Assume $u$ solves

$$\begin{cases} u_{tt} - u_{xx} = f & \text{in } \mathbb{R} \times (0, \infty) \\ u = 0, u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Describe the shape of $u$ for times $t > 3$.

**Proof.** Using Duhamel's principle for the 1D wave equation, we have that the solution to this nonhomogeneous problem is given by

$$u(x,t) = \int_0^t \frac{1}{2} \int_{x-s}^{x+s} f(y, t-s)\, dy\, ds = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s)\, dy\, ds.$$

This formula can be interpreted in a similar manner to the 1D wave equation, where $u(x,t)$ is equal to the integral of $f$ over the triangular region in $\mathbb{R}$ x $(0, \infty)$ defined by the double integral. From the way $f$ is defined, if the entire square $S$ is within this triangle, the integral is zero since the two sections cancel each other out. Additionally, since $S$ is divided by the line $t = x+2$, any region that crosses this line perpendicularly will also integrate to zero since different sections of $f$ cancel out. This happens for triangles where $x < -1$. So, $u(x,t)$ is only nonzero when this triangular region contains only part of $S$, with the separate sections not canceling out.

This happens in a region that is parallel to the line $t = x + 2$ in $\mathbb{R}$ x $(0, \infty)$, since $(x,t)$ in this region contain only part of $S$ in a way that the values of $f$ do not cancel out. In particular, for these $(x,t)$ the triangular region contains more of the positive section, and the closer to the line $t = x + 2$ this edge is, the larger the value of $u(x,t)$. So, this gives a wave along this corridor in $\mathbb{R}$ x $(0, \infty)$ that is one unit wide corresponding to the width of $S$.

So, for times $t > 3$, the shape of $u$ is a mostly flat zero-valued function with a single right-moving wave. This wave is centered at $x = t - 2$ and has width 1.

**3.4**

**(a)** Write down the characteristic equations for the PDE

$$u_t + b \cdot Du = f \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1}$$

where $b \in \mathbb{R}^n$, $f = f(x, t)$.

**Proof.** The characteristic equations are

$$\frac{\partial x_i}{\partial \tau} = b_i, \ \ \frac{\partial t}{\partial \tau} = 1, \ \ \frac{\partial U}{\partial \tau} = f(x, t).$$

**(b)** Use the characteristic ODE to solve (1) subject to the initial condition

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

**Proof.** We will parameterize this initial condition by defining $x_i(0) = a$. Then, the initial conditions are $x_i(0) = a$, $t(0) = 0$, and $U(0) = g(a)$. So, we have three separable PDEs with solutions given by

$$x_i(\tau) = b_i t + a$$

$$t(\tau) = \tau$$

$$U(x_i(\tau), t(\tau)) = U(b_i\tau + a, \tau) = \int_0^\tau f(b_i s + a, s)\, ds + g(a)$$

Using the functions for $x_i$ and $\tau$, we can deduce that $\tau = t$ and $a = x_i - b_i t$. So,

$$\begin{aligned} u(x_i, t) = U(x_i, t) &= \int_0^t f(b_i s + x_i - b_i t, s)\, ds + g(x_i - b_i t) \\ &= \int_0^t f(x_i + (s - t)b_i, s)\, ds + g(x_i - b_i t). \end{aligned}$$

Thus, $u(x, t) = g(x - bt) + \int_0^t f(x + (s - t)b, s)\, ds$.

**3.8** Confirm that the formula $u = g(x - t\mathbf{F}'(u))$ provides an implicit solution for the conservation law

$$u_t + \text{div}\mathbf{F}(u) = 0.$$

**Proof.** We can compute that

$$\begin{aligned} u_t &= -g'(x - t\mathbf{F}'(u))(\mathbf{F}'(u) + t\mathbf{F}''(u)u_t) \\ &= -g'(x - t\mathbf{F}'(u))\mathbf{F}'(u) - g'(x - t\mathbf{F}'(u))t\mathbf{F}''(u)u_t. \end{aligned}$$

Then, solving for $u_t$, we get that

$$u_t = \frac{-g'(x - t\mathbf{F}'(u))\mathbf{F}'(u)}{1 + g'(x - t\mathbf{F}'(u))t\mathbf{F}''(u)}$$

Similarly, to compute $\text{div}\mathbf{F}(u)$, we have that

$$\begin{aligned} u_{x_i} &= g'(x - t\mathbf{F}'(u))(1 - t\mathbf{F}'(u)_{x_i}u_{x_i}) \\ &= g'(x - t\mathbf{F}'(u)) - g'(x - t\mathbf{F}'(u))t\mathbf{F}'(u)_{x_i}u_{x_i}. \end{aligned}$$

Then, solving for $u_{x_i}$, we have that

$$u_{x_i} = \frac{g'(x - t\mathbf{F}'(u))}{1 + g'(x - t\mathbf{F}'(u))t\mathbf{F}'(u)_{x_i}}.$$

So, $\text{div}\mathbf{F}(u) = \mathbf{F}'(u)\sum_{i=1}^{n} u_{x_i}$, meaning that

$$\text{div}\mathbf{F}(u) = \mathbf{F}'(u)\frac{g'(x - t\mathbf{F}'(u))}{1 + g'(x - t\mathbf{F}'(u))t\mathbf{F}''(u)}.$$

So, we have that

$$\begin{aligned} u_t + \text{div}\mathbf{F}(u) &= \frac{-g'(x - t\mathbf{F}'(u))\mathbf{F}'(u)}{1 + g'(x - t\mathbf{F}'(u))t\mathbf{F}''(u)} + \mathbf{F}'(u)\frac{g'(x - t\mathbf{F}'(u))}{1 + g'(x - t\mathbf{F}'(u))t\mathbf{F}''(u)} \\ &= 0. \end{aligned}$$

Thus, the formula $u = g(x - t\mathbf{F}'(u))$ provides an implicit solution to the conservation law.

**4.1** Use separation of variables to find a nontrivial solution $u$ of the PDE

$$u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2} = 0 \text{ in } \mathbb{R}^2.$$

**Proof.** We will look for a solution of the form

$$u(x_1, x_2) = g(x_1)h(x_2).$$

So, $u_{x_1} = g'h, u_{x_2} = gh', u_{x_1x_1} = g''h, u_{x_2x_2} = gh'', u_{x_1x_2} = g'h'$. Thus, we have that

$$(g'h)^2 g''h + 2g'hgh'g'h' + (gh')^2 gh'' = 0.$$

Equivalently, this gives us that

$$(g')^2 g''h^3 + 2(g')^2(h')^2 gh + g^3(h')^2 h'' = 0.$$

Dividing through by $g^3h^3$, we have that

$$\frac{(g')^2 g''}{g^3} + \frac{2(g')^2(h')^2}{g^2h^2} + \frac{(h')^2 h''}{h^3} = 0.$$

We can factor this into two terms as

$$\frac{(g')^2}{g^2}\left(\frac{g''}{g} + \frac{(h')^2}{h^2}\right) + \frac{(h')^2}{h^2}\left(\frac{(g')^2}{g^2} + \frac{h''}{h}\right) = 0.$$

Since we are looking for nontrivial solutions, $\frac{(g')^2}{g^2} \neq 0$ and $\frac{(h')^2}{h^2} \neq 0$. So, to satisfy this above equality we can look for functions $g, h$ such that $\frac{g''}{g} + \frac{(h')^2}{h^2} = 0$ and $\frac{(g')^2}{g^2} + \frac{h''}{h} = 0$. From the first of these equalities, we get that

$$\frac{g''}{g} = -\frac{(h')^2}{h^2} = \mu,$$

so we can separate the variables into two ODEs, $\frac{g''}{g} = \mu$ and $-\frac{(h')^2}{h^2} = \mu$. For now, we will assume $\mu > 0$. Then, we get that $g(x_1) = c_1e^{\sqrt{\mu}x_1} + c_2e^{-\sqrt{\mu}x_1}$ and $h(x_2) = c_3e^{i\sqrt{(\mu)}x_2}$ for constants $c_1, c_2, c_3, \mu$.

Following a similar approach for the other condition $\frac{(g')^2}{g^2} + \frac{h''}{h} = 0$, we have that $\frac{(g')^2}{g^2} = -\frac{h''}{h} = \lambda$. If we assume and therefore $h(x_2) = c_4e^{\sqrt{\lambda}x_2} + c_5e^{-\sqrt{\lambda}x_2}$ and $g(x_1) = c_6e^{i\sqrt{(\lambda)}x_1}$. Notice that if we take $\lambda < 0$, then $g(x_1) = c_6e^{\sqrt{(-\lambda)}x_1}$ and $h(x_2) = c_4e^{i\sqrt{-\lambda}x_2} + c_5e^{-i\sqrt{-\lambda}x_2}$, which are similar to the equations from the first condition. Now, since we want any solution of this this PDE, we can take $\mu = 1$ and $-\lambda = 1$. So, we have that from the first condition,

$$g(x_1) = c_1e^{x_1} + c_2e^{-x_1} \text{ and } h(x_2) = c_3e^{ix_2}$$

while the second condition gives us that

$$g(x_1) = c_6 e^{x_1} \text{ and } h(x_2) = c_4 e^{ix_2} + c_5 e^{-ix_2}.$$

Since we need our solution to satisfy both of these conditions, we need $c_2 = c_5 = 0$, $c_1 = c_6$, and $c_3 = c_4$; then the solution will satisfy the PDE. So, we can take $c_1 = c_6 = 1$, and $c_3 = c_4 = 1$, giving us that $g(x_1) = e^{x_1}$ and $h(x_2) = e^{ix_2}$. Thus,

$$u(x_1, x_2) = g(x_1)h(x_2) = e^{x_1} e^{ix_2} = e^{x_1 + ix_2}$$

is a solution of the PDE.

To check that this is indeed a solution, we note that $u_{x_1} = u_{x_1 x_1} = u$, $u_{x_2} = iu$, $u_{x_2 x_2} = -u$, and $u_{x_1 x_2} = iu$. So, we have that

$$\begin{aligned} u_{x_1}^2 u_{x_1 x_1} + 2u_{x_1} u_{x_2} u_{x_1 x_2} + u_{x_2}^2 u_{x_2 x_2} &= u^2 u + 2u(iu)(iu) + (iu)^2(-u) \\ &= u^3 - 2u^3 + u^3 \\ &= 0. \end{aligned}$$

Thus, $u(x_1, x_2) = e^{x_1 + ix_2}$ is a solution of the PDE.

**4.7** Consider the viscous conservation law

$$u_t + F(u)_x - au_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty), \tag{2}$$

where $a > 0$ and $F$ is uniformly convex.

**(a)** Show $u$ solves (2) if $u(x,t) = v(x - \sigma t)$ and $v$ is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} dz \ (s \in \mathbb{R}),$$

where $b$ and $c$ are constants.

**Proof.** For $u(x,t) = v(x - \sigma t)$, (2) becomes

$$-\sigma v' + F(v)v' - au_{xx} = 0.$$

From this, we can see that $-\sigma v + F(v) - av'$ must be constant, so $-\sigma v + F(v) - av' = -b$ for some constant $b$. So, $v' = \frac{-\sigma v + F(v) + b}{a}$. We can rearrange this equation as

$$\frac{1}{v'} = \frac{a}{-\sigma v + F(v) + b}.$$

Then, we can integrate both sides from $v(0)$ to $v(s)$ to get that

$$\int_{v(0)}^{v(s)} \frac{ds}{dv} dv = \int_{v(0)}^{v(s)} \frac{a}{F(z) - \sigma z + b} dz,$$

where $\int_{v(0)}^{v(s)} \frac{ds}{dv} dv = s - 0 = s$. Thus, we get the implicit formula for $s$ defined by

$$s = \int_{v(0)}^{v(s)} \frac{a}{F(z) - \sigma z + b} dz.$$

So, $v(0) = c$ as some constant, we have that

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} dz$$

is implicit formula for constants $b$ and $c$ that provides a solution to (2).

**(b)** Demonstrate that we can find a traveling wave satisfying

$$\lim_{s\to-\infty} v(s) = u_l,\ \lim_{s\to\infty} v(s) = u_r$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

**Proof.** ($\rightarrow$) Suppose $\lim_{s\to-\infty} v(s) = u_l$ and $\lim_{s\to\infty} v(s) = u_r$. This means that we have horizontal asymptotes, so $\lim_{s\to\pm\infty} v'(s) = 0$. From (a), we found that $-\sigma v + F(v) - av' = -b$. So, as $s \to \infty$, this gives us that $-\sigma u_r + F(u_r) = -b$. Similarly, as $s \to -\infty$, this gives us that $-\sigma u_l + F(u_l) = -b$. Combining these two equations, we have that $-\sigma u_r + F(u_r) = -\sigma u_l + F(u_l)$ or equivalently,

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

($\leftarrow$) Suppose $\sigma = \frac{F(u_l)-F(u_r)}{u_l-u_r}$. To show that we can find a traveling wave solution, $v$, satisfying

$$\lim_{s\to-\infty} v(s) = u_l,\ \lim_{s\to\infty} v(s) = u_r,$$

we must first consider the first-order ODE from part (a) and show that a profile $v$ exists that connects an unstable equilibrium to a stable one. From part (a), we have that a traveling wave solution must satisfy that $v' = \frac{-\frac{F(u_l)-F(u_r)}{u_l-u_r}v + F(v) + b}{a}$ for some constant $b$. Rearranging, we get the following first-order ODE:

$$av' = -\frac{F(u_l) - F(u_r)}{u_l - u_r}v + F(v) + b.$$

The equilibria of this ODE occur when $-\frac{F(u_l)-F(u_r)}{u_l-u_r}v + F(v) + b = 0$. We can rearrange this as $F(v) = \frac{F(u_l)-F(u_r)}{u_l-u_r}v - b$ so that the right-hand side of the equation is linear with a slope equal to the average slope of $F$ between $u_r$ to $u_l$. In order for a traveling wave solution to exist with the desired limits, we need the two equilibria to be $u_l$ and $u_r$, with $u_l > u_r$. To check that such a wave exists, we will first choose $b$ so that $u_r$ is an equilibrium and then show that $u_l$ is also an equilibrium.

So, if we choose $b$ appropriately so that $u_r$ is an equilibrium, we have that

$$F(u_r) = \frac{F(u_l) - F(u_r)}{u_l - u_r}u_r - b,$$

so $b = \frac{F(u_l)-F(u_r)}{u_l-u_r}u_r - F(u_r)$. Substituting this value of $b$ back into the the equilibrium condition, we get that

$$F(v) = \frac{F(u_l) - F(u_r)}{u_l - u_r} v - \frac{F(u_l) - F(u_r)}{u_l - u_r} u_r - F(u_r)$$
$$= \frac{F(u_l) - F(u_r)}{u_l - u_r} (v - u_r) - F(u_r).$$

So, this implies that if $v$ satisfies

$$\frac{F(v) - F(u_r)}{v - u_r} = \frac{F(u_l) - F(u_r)}{u_l - u_r},$$

then $v$ is an equilibrium. From this equality, we see that $u_l$ is also an equilibrium for this choice of $b$. So, if we choose $b$ so that $u_r$ is an equilibrium, $u_l$ is automatically an equilibrium as well.

Now, we must show that in this case, the traveling wave satisfies the limit conditions. Since $F$ is uniformly convex, $F(v) < \frac{F(u_l)-F(u_r)}{u_l-u_r} v - b$ for all $v \in (u_r, u_l)$ and $F''(v) > 0$ for all $v$. This means that for $u_r < v < u_l$, $\frac{F(u_l)-F(u_r)}{u_l-u_r} v - b > F(v)$, while for $v < u_r$ or $v > u_l$, $\frac{F(u_l)-F(u_r)}{u_l-u_r} v - b < F(v)$. Then, since $a > 0$, this implies that $v' < 0$ for $u_r < v < u_l$ and $v' > 0$ for $v < u_r$ or $v > u_l$. Thus, $v = u_r$ is a stable equilibrium and $v = u_l$ is an unstable equilibrium for this ODE.

So, with this choice of $b$, we can find a solution, $v$, that connects $u_l$, which is an unstable equilibrium of the ODE, to $u_r$, a stable equilibrium of the ODE. With any initial condition, $v_0$, such that $u_r < v_0 < u_l$, the solution goes to $u_r$ as $t \to \infty$ and goes to $u_l$ as $t \to -\infty$. Thus, we can find a traveling wave solution such that

$$\lim_{s \to -\infty} v(s) = u_l, \ \lim_{s \to \infty} v(s) = u_r.$$

**(c)** Let $u^\epsilon$ denote the above traveling wave solution of (2) for $a = \epsilon$, with $u^\epsilon(0,0) = \frac{u_l - u_r}{2}$. Compute $\lim_{\epsilon \to 0} u^\epsilon$ and explain your answer.

**Proof.** From the integral,

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} dz$$

with

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r},$$

we consider $v(s) \in (u_r, u_l)$. As $v(s) \to u_l$ or $v(s) \to u_r$, the denominator goes to 0, meaning that $s \to \mp\infty$. So, we can fix $s \neq 0$ in this formula, substituting $a = \epsilon$. So, we have

$$\frac{s}{\epsilon} = \int_c^{u^\epsilon(s)} \frac{1}{F(z) - \sigma z + b} dz.$$

Taking the limit as $\epsilon \to 0$, the LHS goes to $\pm\infty$ depending on the sign of $s$, so on the RHS, $u^\epsilon \to u_r$ or $u^\epsilon \to u_l$ depending on the sign of $s$. So, as $\epsilon \to 0$, this solution $u^\epsilon$ converges to our implicit traveling wave solution $v$ from parts (a) and (b) with initial condition $u^\epsilon(0,0) = \frac{u_l + u_r}{2}$.

**Extra Problems**

**1.** Apply separation of variables to the Telegraph equation, pg 4, to find solutions that are bounded for all $x$ and all positive $t$.

**Proof.** The telegraph equation is given by

$$u_{tt} + 2du_t - u_{xx} = 0.$$

We will apply separation of variables to look for a solution of the form $u(x,t) = g(x)h(t)$. With this form, the telegraph equation becomes $gh'' + 2dgh' - g''h = 0$. By factoring out $g$ from the first two terms and rearranging the equation, we get that

$$\frac{h'' + 2dh'}{h} = \frac{g''}{g} = \mu$$

for some constant $\mu$. These equations are equal to some constant because the equality holds for all $(x,t)$. Using this fact, we can separate this equation into two ODEs, $g'' = \mu g$ and $h'' + 2dh' = \mu h$.

For $g'' = \mu g$, we get that $g(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$ for some constants $c_1, c_2$. For $h'' + 2dh' = \mu h$, we can write out the characteristic polynomial $m^2 + 2dm - \mu = 0$ and get that $m = -d \pm \sqrt{d^2 + \mu}$. So, $h(t) = c_3 e^{-dt + t\sqrt{d^2+\mu}} + c_4 e^{-dt - t\sqrt{d^2+\mu}}$ for some constants $c_3, c_4$.

Thus, we have solutions to the telegraph equation given by

$$u(x,t) = g(x)h(t) = (c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x})(c_3 e^{-dt+t\sqrt{d^2+\mu}} + c_4 e^{-dt-t\sqrt{d^2+\mu}}).$$

These solutions are bounded for all $x$ and positive $t$. Taking $\mu, c_1, c_2, c_3, c_4 = 1$, we can obtain an explicit bounded solution to the Telegraph equation

$$u(x,t) = e^{x-dt+t\sqrt{d^2+1}} + e^{-x-dt+t\sqrt{d^2+1}} + e^{x-dt-t\sqrt{d^2+1}} + e^{-x-dt-t\sqrt{d^2+1}}.$$

**2.** Use the Fourier transform to solve the Telegraph equation with initial data $u(x;0) = f(x)$ and $u_t(x;0) = g(x)$. What must you assume about $f, g$?

**Proof.** As in the previous problem, the telegraph equation is given by

$$u_{tt} + 2du_t - u_{xx} = 0.$$

In order to use the Fourier Transform to solve this equation, we must assume that we can take the Fourier Transform of the initial conditions. So, we must assume that $f, g \in L^2(\mathbb{R}^n$. Then, taking the Fourier Transform with respect to the spatial variables, we get that

$$\hat{u}_{tt} + 2d\hat{u}_t + |y|^2\hat{u} = 0$$

with initial conditions $\hat{u} = \hat{f}$ and $\hat{u}_t = \hat{g}$. We can solve this ODE using the characteristic polynomial $m^2 + 2dm + |y|^2 = 0$, giving us that

$$m = \frac{-2d \pm \sqrt{4d^2 - 4|y|^2}}{2} = -d \pm \sqrt{d^2 - |y|^2}.$$

Since $y$ is a variable, this value may be real or complex depending on $y$. So, we must consider both of these cases, meaning that the general solution of this ODE is

$$\hat{u}(y,t) = \begin{cases} c_1 e^{(-d+\sqrt{d^2-|y|^2})t} + c_2 e^{(-d-\sqrt{d^2-|y|^2})t} \text{ for } |y| \leq d \\ c_1 e^{(-d+i\sqrt{d^2-|y|^2})t} + c_2 e^{(-d-i\sqrt{d^2-|y|^2})t} \text{ for } |y| > d \end{cases}$$

for some constants $c_1$ and $c_2$. To enforce the initial conditions, we choose $c_1$ and $c_2$ to satisfy that

$$\hat{f} = c_1 + c_2$$

so that $\hat{u} = \hat{f}$ at $t = 0$. Additionally, we need $\hat{u}_t = \hat{g}$, so differentiating $\hat{u}$ with respect to $t$ and setting $t = 0$, we get that

$$\hat{u}_t(y,0) = \hat{h}(y) = \begin{cases} c_1(-d + \sqrt{d^2 - |y|^2}) + c_2(-d - \sqrt{d^2 - |y|^2}) \text{ for } |y| \leq d \\ c_1(-d + i\sqrt{d^2 - |y|^2}) + c_2(-d - i\sqrt{d^2 - |y|^2}) \text{ for } |y| > d \end{cases}$$

So, by choosing $c_1$ and $c_2$ to satisfy these conditions, we can then take the inverse Fourier Transform of both sides and obtain that

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{1/2}} \int_{\{|y| \leq d\}} e^{ixy} c_1 e^{(-d+\sqrt{d^2-|y|^2})t} + e^{ixy} c_2 e^{(-d-\sqrt{d^2-|y|^2})t} \\ &\quad + \frac{1}{(2\pi)^{1/2}} \int_{\{|y| > d\}} e^{ixy} c_1 e^{(-d+i\sqrt{d^2-|y|^2})t} + e^{ixy} c_2 e^{(-d-i\sqrt{d^2-|y|^2})t} \\ &= \frac{e^{-dt}}{(2\pi)^{1/2}} \int_{\{|y| \leq d\}} c_1 e^{ixy+t\sqrt{d^2-|y|^2}} + c_2 e^{ixy+t\sqrt{d^2-|y|^2}} \\ &\quad + \frac{e^{-dt}}{(2\pi)^{1/2}} \int_{\{|y| > d\}} c_1 e^{ixy+it\sqrt{d^2-|y|^2}} + c_2 e^{ixy-it\sqrt{d^2-|y|^2}}. \end{aligned}$$

This is a solution to the Telegraph equation using the Fourier Transform.