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APM 501: Differential Equations I – Portfolio Project

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Problem 1

Find all upper Jordan canonical forms of 7x7 real matrices with the one eigenvalue $\lambda = 5$ having multiplicity $n = 7$. In each case, give the values of all the deficiency indexes (δ_1,δ_2,...) and the block-size count ν_1,ν_2,...

Solution

For 7x7 matrices with $\lambda = 5$ as an eigenvalue with multiplicity $n = 7$, there are 15 different matrices in upper Jordan canonical form. Each is unique up to the organization of the blocks in the matrix. Note that even if the blocks in the Jordan canonical form are organized in a different order, the deficiency indicies and the block-size counts will be the same. So, in each case below, the deficiency indices and block-size count are given along with a sample matrix with the blocks organized from largest to smallest.

1. The Jordan form has seven 1x1 blocks.

$$\begin{array}{ll} \delta_1 = 7 & \nu_1 = 7 \\ \delta_2 = 7 & \nu_2 = 0 \\ \delta_3 = 7 & \nu_3 = 0 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

2. The Jordan form has one 2x2 block and five 1x1 blocks.

$$\begin{array}{ll} \delta_1 = 6 & \nu_1 = 5 \\ \delta_2 = 7 & \nu_2 = 1 \\ \delta_3 = 7 & \nu_3 = 0 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

3. The Jordan form has one 3x3 block and four 1x1 blocks.

$$\begin{array}{ll} \delta_1 = 5 & \nu_1 = 4 \\ \delta_2 = 6 & \nu_2 = 0 \\ \delta_3 = 7 & \nu_3 = 1 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

4. The Jordan form has two 2x2 blocks and three 1x1 blocks.

$$\begin{array}{ll} \delta_1 = 5 & \nu_1 = 3 \\ \delta_2 = 7 & \nu_2 = 2 \\ \delta_3 = 7 & \nu_3 = 0 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

5. The Jordan form has one 4x4 block and three 1x1 blocks.

$$\begin{array}{ll} \delta_1 = 4 & \nu_1 = 3 \\ \delta_2 = 5 & \nu_2 = 0 \\ \delta_3 = 6 & \nu_3 = 0 \\ \delta_4 = 7 & \nu_4 = 1 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

6. The Jordan form has one 3x3 block, one 2x2 block and two 1x1 blocks.

$$\begin{array}{ll} \delta_1 = 4 & \nu_1 = 2 \\ \delta_2 = 6 & \nu_2 = 1 \\ \delta_3 = 7 & \nu_3 = 1 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

7. The Jordan form has three 2x2 blocks and one 1x1 block.

$$\begin{array}{ll} \delta_1 = 4 & \nu_1 = 1 \\ \delta_2 = 7 & \nu_2 = 3 \\ \delta_3 = 7 & \nu_3 = 0 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

8. The Jordan form has one 5x5 block and two 1x1 blocks.

$$\begin{matrix} \delta_1 = 3 & \nu_1 = 2 \\ \delta_2 = 4 & \nu_2 = 0 \\ \delta_3 = 5 & \nu_3 = 0 \\ \delta_4 = 6 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 1 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{matrix} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

9. The Jordan form has one 4x4 block, one 2x2 block, and one 1x1 block.

$$\begin{matrix} \delta_1 = 3 & \nu_1 = 1 \\ \delta_2 = 5 & \nu_2 = 1 \\ \delta_3 = 6 & \nu_3 = 0 \\ \delta_4 = 7 & \nu_4 = 1 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{matrix} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

10. The Jordan form has two 3x3 blocks and one 1x1 block.

$$\begin{matrix} \delta_1 = 3 & \nu_1 = 1 \\ \delta_2 = 5 & \nu_2 = 0 \\ \delta_3 = 7 & \nu_3 = 2 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{matrix} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

11. The Jordan form has one 3x3 block and two 2x2 blocks.

$$\begin{matrix} \delta_1 = 3 & \nu_1 = 0 \\ \delta_2 = 6 & \nu_2 = 2 \\ \delta_3 = 7 & \nu_3 = 1 \\ \delta_4 = 7 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{matrix} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

12. The Jordan form has one 6x6 block and one 1x1 block.

$$\begin{matrix} \delta_1 = 2 & \nu_1 = 1 \\ \delta_2 = 3 & \nu_2 = 0 \\ \delta_3 = 4 & \nu_3 = 0 \\ \delta_4 = 5 & \nu_4 = 0 \\ \delta_5 = 6 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 1 \\ \delta_7 = 7 & \nu_7 = 0 \end{matrix} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

13. The Jordan form has one 5x5 block and one 2x2 block.

$$\begin{array}{ll} \delta_1 = 2 & \nu_1 = 0 \\ \delta_2 = 4 & \nu_2 = 1 \\ \delta_3 = 5 & \nu_3 = 0 \\ \delta_4 = 6 & \nu_4 = 0 \\ \delta_5 = 7 & \nu_5 = 1 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

14. The Jordan form has one 4x4 block and one 3x3 block.

$$\begin{array}{ll} \delta_1 = 2 & \nu_1 = 0 \\ \delta_2 = 4 & \nu_2 = 0 \\ \delta_3 = 6 & \nu_3 = 1 \\ \delta_4 = 7 & \nu_4 = 1 \\ \delta_5 = 7 & \nu_5 = 0 \\ \delta_6 = 7 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 0 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

15. The Jordan form has one 7x7 block.

$$\begin{array}{ll} \delta_1 = 1 & \nu_1 = 0 \\ \delta_2 = 2 & \nu_2 = 0 \\ \delta_3 = 3 & \nu_3 = 0 \\ \delta_4 = 4 & \nu_4 = 0 \\ \delta_5 = 5 & \nu_5 = 0 \\ \delta_6 = 6 & \nu_6 = 0 \\ \delta_7 = 7 & \nu_7 = 1 \end{array} \quad \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Problem 2

Find two real 8x8 real matrices A and B that have a quadruple eigenvalue $\lambda = 2 + i$, with $\dim E^{\lambda} = 2$, and $w_1 = (e_1 + e_3) + i(e_2 + e_8)$ and $w_2 = (e_2 + e_7) + i(e_4 + e_8)$ as eigenvectors. Moreover, A should have two generalized eigenvectors w_3 and w_4 such that $(A - \lambda I)w_3$ is a multiple of $iw_1 - (2 + i)w_2$ and $(A - \lambda I)w_4$ is a multiple of w_3, while B should have two generalized eigenvectors w_3 and w_4 such that $(A - \lambda I)w_3$ is a multiple of $(2 + i)w_1 - iw_2$ and $(A - \lambda I)w_4$ is a multiple of $iw_1 - (2 + i)w_2$. Here $e_1, ..., e_8$ denotes the standard basis of R^8.

Solution

To find matrices corresponding to the required eigenvalues and eigenvectors, I utilized the upper Jordan canonical forms of the matrices. For A, neither of the generalized eigenvectors were found from a cycle starting from the original eigenvectors, so neither w_1 nor w_2 started the cycle that led to w_3 and w_4. The generalized eigenvector w_3 is generated by a combination of w_1 and w_2, so this combination starts one cycle. To have an eigenvector that starts the cycle, the eigenvectors had to redefined in such a way so that one of them starts the cycle and these eigenvectors still span the same space in R^8. Thus, new eigenvectors, w_1^* and w_2^* were defined so that w_2 started the cycle leading to w_3 and w_4. By doing this, the Jordan canonical form could be created with one 2x2 block and one 6x6 block (where the complex conjugate pair of eigenvalues is a 2x2 block) since the new w_2 starts the cycle that leads to the generalized eigenvectors w_3 and w_4. Thus the Jordan canonical form is given by

$$J = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Along the diagonal, we have the 4 2x2 blocks corresponding to the pair of conjugate eigenvalues $2 \pm i$ with multiplicity 4. After redefining the eigenvectors, w_2^* begins a cycle that leads to the generalized eigenvectors w_3 and w_4, so the Jordan matrix is also broken up into two blocks, one 2x2 block and one 6x6 block with identity matrices in the superdiagonal representing the cycle that leads to the generalized eigenvectors.

This Jordan canonical form can be derived from the information given based on the cycle of eigenvectors. The Jordan canonical form satisfies the the equality $AP = PJ$. First, $(A - \lambda I)w_3$ is a multiple of $w_2^* = iw_1 - (2+i)w_2$, so $(A - \lambda I)w_3 - nw_2^* = 0$ for some $n \in R$. However, since we can choose w_3 to satisfy this equation, we can choose $n = 1$. So, $(A - \lambda I)w_3 - w_2^* = 0$, or equivalently, $Aw_3 = \lambda w_3 + w_2^*$. In the equation $AP = PJ$, Aw_3 corresponds to columns 5 and 6 of AP, so we need $\lambda w_3 + w_2^*$ to be in columns 5 and 6 of PJ. Since w_2^* is located in columns 3 and 4 of P and w_3 is in columns 5 and 6 of P, we need columns 5 and 6 of J to give $1 * w_2^* + (2+i)w_3$, which is done by placing a 2x2 identity matrix in the super diagonal and the 2x2 matrix corresponding to the complex conjugate pair of eigenvalues $2 \pm i$ in the diagonal. This provides the desired equality. This process is repeated similarly for w_4, in which $(A - \lambda I)w_4 - w_3 = 0$.

With the Jordan form, all that remained was to create the eigenvector matrix. With only the requirement now being that the generalized eigenvectors are created in specific cycles, these remaining generalized eigenvectors can be chosen in any way so that they are linearly independent of all other eigenvectors and generalized eigenvectors. Then, they span all of R^8, meaning that the eigenvalue $\lambda = 2 + i$ will have multiplicity 4. So, eigenvectors were chosen to satisfy this and form the matrix P needed to produce A. Then, A was calculated using $A = PJP^{-1}$, where J is the Jordan canonical form of A found using the eigenvalues and the cycles.

The redefined eigenvectors, w_1^* and w_2^*, were chosen as $w_1^* = (2+i)w_1 - iw_2$ and $w_2^* = iw_1 - (2+i)w_2$ so that they are linearly independent. The generalized eigenvectors chosen for w_3 and w_4 were $w_3 = e_5 + ie_1$ and $w_4 = e_7 + ie_6$. These were found by looking at the span of the eigenvectors w_1 and w_2 and finding linearly independent vectors: e_1, e_5, e_6, and e_7.

$$A = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\ -1 & 2 & 0 & 1 & -3 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ -2 & -1 & 2 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 & -1 \\ -1 & 0 & 1 & 1 & -2 & 1 & 2 & 0 \\ -2 & -1 & 1 & -1 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Finding the matrix B was a similar process, but instead of one eigenvector starting a cycle that leads to two generalized eigenvectors, w_3 and w_4, two different eigenvectors start two different cycles. So, w_1^* and w_2^* were again redefined as linear combinations of the original eigenvectors so that each start a cycle. The

Jordan canonical form in this case was two blocks of size 4x4 for these two cycles of length 2 with eigenvalues that are complex conjugate pairs. Thus the Jordan canonical form is given by

$$J = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Along the diagonal, we have the 4 2x2 blocks corresponding to the pair of conjugate eigenvalues $2 \pm i$ with multiplicity 4. After redefining the eigenvectors, w_2^* begins a cycle that leads to the generalized eigenvectors w_3 and w_4, so the Jordan matrix is broken up into two blocks, one 2x2 block and one 6x6 block with identity matrices in the superdiagonal representing the cycle.

As before, the Jordan canonical form can be derived from the information given based on the cycle of eigenvectors. The Jordan canonical form satisfies the the equality $AP = PJ$. So, $(A - \lambda I)w_3$ is a multiple of w_1^*. Again choosing the multiple of the eigenvalue to be 1, we get that $(A - \lambda I)w_3 - w_1^* = 0$, or equivalently, $Aw_3 = \lambda w_3 + w_1^*$. This time, because the cycles are different, we reorganize the eigenvector matrix so that in the equation $AP = PJ$, Aw_3 corresponds to columns 3 and 4 of AP, so we need $\lambda w_3 + w_1^*$ to be in columns 3 and 4 of PJ. Since w_1^* is located in columns 1 and 2 of P and w_3 is in columns 3 and 4 of P, we need columns 3 and 4 of J to give $1 * w_1^* + (2 + i)w_3$, which is done by placing a 2x2 identity matrix in the super diagonal and the 2x2 matrix corresponding to the complex conjugate pair of eigenvalues $2 \pm i$ in the diagonal. This provides the desired equality. As before, this process is repeated similarly for w_4, in which $(A - \lambda I)w_4$ is a multiple of w_2^*.

Once w_1^* and w_2^* were defined and the new Jordan canonical form created, the generalized eigenvectors were chosen to be linearly independent of the rest so that the set of eigenvectors and generalized eigenvectors span R^8 and P is invertible. Then, B was calculated using $B = PJP^{-1}$, where J is the Jordan canonical form of B found using the eigenvalues and the cycles.

The redefined eigenvectors, w_1^* and w_2^*, were chosen as $w_1^* = (2 + i)w_1 - iw_2$ and $w_2^* = iw_1 - (2 + i)w_2$ according to the necessary cycles. The generalized eigenvectors chosen for w_3 and w_4 were $w_3 = e_5 + ie_1$ and $w_4 = e_7 + ie_6$, using the same linearly independent vectors found for A.

$$B = \begin{bmatrix} 3 & 0 & -1 & -1 & 1 & 1 & 0 & 1 \\ 1 & 5 & -2 & 4 & -1 & -1 & -3 & -3 \\ 1 & 0 & 1 & -1 & 2 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & -2 & 1 & 2 \\ 1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 & -1 \\ -1 & 2 & 1 & 3 & 0 & 0 & 0 & -2 \\ 2 & -1 & -3 & -1 & 0 & -2 & 0 & 3 \end{bmatrix}$$

Problem 3

Using the matrix exponential e^{At}, find the solution of the initial value problem
$\dot{X} = AX, X(0) = [-1, 1, -1, 1, -1, 1]^T$,
where

$$A = \begin{bmatrix} -1 & 11 & 21.5 & 6 & -25.5 & -19 \\ 49 & -71 & -218.5 & -106 & 188.5 & 196 \\ 5 & -8 & -22 & -9 & 21 & 22 \\ 10 & -11 & -42 & -20 & 32 & 36 \\ 21 & -28 & -85 & -43 & 74 & 78 \\ 9 & -17 & -51.5 & -20 & 45.5 & 46 \end{bmatrix}$$

Solution

To solve problem 3, I used MATLAB to help calculate the matrix exponential. First, the eigenvalues and eigenvectors of A were calculated and used to create the Jordan canonical form of A, along with P and P^{-1}. The matrix A only has two eigenvalues, which are the complex conjugate pair $1 \pm 3i$, and one corresponding eigenvector, w_1. Thus, to create the Jordan canonical form, there had to be two more generalized eigenvectors found through cycles. This gives the following Jordan canonical form

$$J = \begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}$$

since one eigenvector starts the cycle that leads to two generalized eigenvectors. In this case, the generalized eigenvectors, w_2 and w_3 would be found so that $(A - (1 + 3i)I)w_2 = w_1$ and $(A - (1 + 3i)I)w_3 = w_2$. This can again be seen in the equation $AP = PJ$. For example, Aw_2 is the third and fourth columns of AP, and from the above equation, $Aw_2 = w_1 + (1 + 3i)w_2$, where w_1 and w_2 are in the columns of P. Thus, columns 3 and 4 of J are designed in such a way so that in PJ, columns 3 and 4 give us $w_1 + (1+3i)w_2$. This is done with an identity matrix in the superdiagonal (to give $1 * w_1$), and the eigenvalue 2x2 block in the diagonal (to give $(1+3i)w_2$). Thus, the Jordan matrix J was derived from just the eigenvalues and eigenvectors of A.

With the Jordan form, MATLAB was used to find the generalized eigenvectors, which had to be interpreted and rewritten to match the above Jordan form. The generalized eignvectors had complex entries which had to split into the real and imaginary parts for the form used in class. After this process was completed, the result was verified using the equation $A = PJP^{-1}$, where P is the eigenvector matrix and J is the Jordan canonical form of A.

With this structure, the matrix exponential is easier to calculate since $A = PJP^{-1}$, and $e^A = Pe^J P^{-1}$, with $e^{tA} = Pe^{tJ}P^{-1}$. To calculate the matrix exponential of the Jordan form, known formulas from the textbook were applied. Then, $e^{tA} = Pe^{tJ}P^{-1}$ and the solution to the initial value problem is $e^{tA}x_0$ (Note that P^{-1} was giving large decimals in MATLAB and had to be rounded to get accessible answers). Doing these calculations gives the following solution:

$$X(t) = \begin{bmatrix} e^t(\sin 3t - \cos 3t) + te^t(\cos 3t - \sin 3t) + t^2e^t(\frac{1}{2}\cos 3t + \frac{11}{10}\sin 3t) \\ e^t(\cos 3t - \sin 3t) + te^t(2\cos 3t + 3\sin 3t) + t^2e^t(\frac{1}{2}\cos 3t - \frac{9}{10}\sin 3t) \\ e^t(\sin 3t - \cos 3t) + te^t(\sin 3t - \cos 3t) - t^2e^t(\frac{1}{2}\cos 3t + \frac{1}{10}\sin 3t) \\ e^t(\sin 3t + \cos 3t) + te^t(\cos 3t + \sin 3t) + t^2e^t(\cos 3t + \sin 3t) \\ -e^t(\sin 3t - \cos 3t) + te^t(\cos 3t + \sin 3t) - t^2e^t(\frac{1}{2}\cos 3t - \frac{7}{10}\sin 3t) \\ e^t(2\sin 3t + \cos 3t) + te^t(\cos 3t + 2\sin 3t) + t^2e^t(\frac{1}{2}\cos 3t + \frac{7}{10}\sin 3t) \end{bmatrix}$$