School of Mathematical and Statistical Sciences
Arizona State University

Student: Lee Reeves

Course: APM 501

Program: Mathematics MA

Instructor: Fabio Milner

Date: Fall 2019

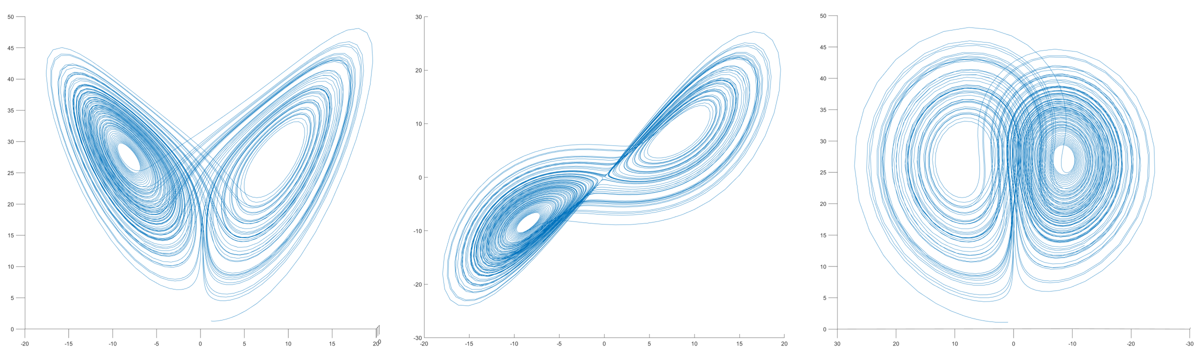
APM 501 Master's Portfolio Project
Characterizing the Lorenz System

Lee Reeves

January 24, 2020

Abstract

In this portfolio project, I will apply the techniques developed by Lawrence Perko (Perko 2001) and taught in APM 501 at Arizona State University to the Lorenz system, a famous system of nonlinear ordinary differential equations introduced by Edward N. Lorenz in 1963, that, for some parameter values, exhibits chaotic behavior, i.e. strong sensitivity to initial conditions, or as Lorenz described it: "the present determines the future, but the approximate present does not approximately determine the future."



Front view (along x_2 axis) Top view (along x_3 axis) Side view (along x_1 axis)

Figure 1: Three perspectives on a solution of the Lorenz system with $\sigma = 10, r = 28, b = 8/3$ starting from $(1, 1, 1)^T$ for t from 0 to 100. MATLAB source code is in the appendix.

Introduction

In 1963, Edward M. Lorenz simplified Saltzman's equations for convection (limited to a vertical plane) in a fluid of uniform depth, when the temperature difference between the upper and lower surfaces is held constant, and obtained the following system of nonlinear ordinary differential equations (Lorenz 1963):

$$
\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\
\dot{x}_3 &= x_1 x_2 - b x_3
\end{aligned}
\tag{1}
$$

In this system, the variables x_1, x_2, and x_3 do not refer to coordinates in space. Instead, as Lorenz described them, x_1 is proportional to the intensity of the convective motion, x_2 is proportional to the temperature difference between the ascending and descending currents, and x_3 is proportional to the distortion of the vertical temperature profile from linearity.

The parameters σ, r, and b are positive constants; σ is the Prandtl number, r is the ratio of the Rayleigh numbers of the ascending and descending currents, and b is a constant related to the given space.

These equations also describe the behavior of a waterwheel with leaky cups, a model of the Lorenz system reportedly invented by Willem Markus and described in detail in chapter 9 of Strogatz 2015. Several people have made YouTube videos showing this model in action.

Simple Properties

Symmetry

The Lorenz equations are symmetric under the transformation $(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$, so any solution must have a counterpart under the same symmetry, which contributes to the famous butterfly shape of the Lorenz attractor.

Volume

The divergence of the Lorenz system is

$$\nabla \cdot f = \frac{\partial}{\partial x_1}[\sigma(x_2 - x_1)] + \frac{\partial}{\partial x_2}[-x_1 x_3 + r x_1 - x_2] + \frac{\partial}{\partial x_3}[x_1 x_2 - b x_3] = -\sigma - 1 - b < 0$$

so the Lorenz system contracts volumes in phase space.(Strogatz 2015)

Existence and Uniqueness of Solutions

Following Perko's notation, we may also write the Lorenz system as

$$\mathbf{X} = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \tag{2}$$

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}) = \begin{bmatrix} \sigma(x_2 - x_1) \\ x_1(r - x_3) - x_2 \\ x_1 x_2 - b x_3 \end{bmatrix} \tag{3}$$

The function $\mathbf{f}$ is differentiable on $\mathbb{R}^3$ for all values of $\sigma, r, b \in \mathbb{R}$, and the derivative of $\mathbf{f}$ is the Jacobian matrix:

$$Df = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - x_3 & -1 & -x_1 \\ x_2 & x_1 & -b \end{bmatrix} \tag{4}$$

which is continuous on $\mathbb{R}^3$ for all values of $\sigma, r, b \in \mathbb{R}$, so $f \in C^1(\mathbb{R})$.

Therefore, by the fundamental-existence and uniqueness theorem, there exists an interval $(-a, a)$ on which this system has a unique solution. But we don't have a closed form solution for $\mathbf{X}$, so we cannot find $\lim_{t\to\beta^-} \mathbf{X}(t)$ to determine a maximal interval of existence (α, β).

Instead, we will have to take a different approach, showing that there exists a trapping region in which all trajectories end, and therefore the right maximal interval of existence of (1) is $[0, \infty)$. To do so, we begin by defining a Lyapunov function for (1).

Trapping Region

Let ϕ_t be the flow of the differential equation (1). We define

$$V(\mathbf{X}) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$$

Then $V \in C^1(\mathbb{R})$ and by definition 2 in section 2.9 the derivative of $V(\mathbf{X})$ along the solution $\phi_t(\mathbf{X})$ is

$$\begin{aligned}
\dot{V}(\mathbf{X}) &= DV(\mathbf{X})\mathbf{f}(\mathbf{X}) \\
&= \begin{bmatrix} 2rx_1 & 2\sigma x_2 & 2\sigma x_3 - 4\sigma r \end{bmatrix} \begin{bmatrix} \sigma(x_2 - x_1) \\ x_1(r - x_3) - x_2 \\ x_1x_2 - bx_3 \end{bmatrix} \\
&= 2\sigma r x_1(x_2 - x_1) + 2\sigma x_2[x_1(r - x_3) - x_2] + 2\sigma(x_3 - 2r)[x_1x_2 - bx_3] \\
&= 2\sigma[rx_1x_2 - rx_1^2 + rx_1x_2 - x_1x_2x_3 - x_2^2 + x_1x_2x_3 - bx_3^2 - 2rx_1x_2 + 2brx_3] \\
&= -2\sigma[rx_1^2 + x_2^2 + bx_3^2 - 2brx_3] \\
&= -2\sigma[rx_1^2 + x_2^2 + b(x_3 - r)^2 - br^2].
\end{aligned}$$

Since $\sigma > 0$, $\dot{V}(\mathbf{X}) > 0$ only when $rx_1^2 + x_2^2 + b(x_3 - r)^2 - br^2 < 0$, or equivalently, when

$$\frac{x_1^2}{br} + \frac{x_2^2}{br^2} + \frac{(x_3 - r)^2}{r^2} < 1,$$

which is the equation for an ellipsoid E with center $(0, 0, r)^T$ and the positive endpoints of the principal axes being $(\sqrt{br}, 0, r)$, $(0, \sqrt{br^2}, r)$, and $(0, 0, 2r)$.

On the surface of E, and only on the surface of E, $\dot{V}(\mathbf{X}) = 0$, so any minima of V must be found on the surface of E. This includes the global minimum $V((0, 0, 2r)) = 0$.

Outside of E, $\dot{V}(\mathbf{X}) < 0$, so any solution that begins outside the ellipsoid must approach a minimum, and therefore must approach the ellipsoid.

Inside of E, $\dot{V}(\mathbf{X}) > 0$, so any solution that begins inside, or at any time enters, the ellipsoid must approach a minimum on the ellipsoid if time is reversed, and must be repelled from the ellipsoid as time moves forward, and therefore must remain inside the ellipsoid as t increases.

Therefore all solutions are bounded, and exist for all $t > 0$.

Critical Points

The system (2) has critical (or equilibrium) points when $f(\mathbf{X}) = 0$, that is:

$$\begin{aligned} \sigma(x_2 - x_1) &= 0 \\ x_1(r - x_3) - x_2 &= 0 \\ x_1x_2 - bx_3 &= 0 \end{aligned}$$

which implies that $x_2 = x_1$ and $x_3 = \frac{1}{b}x_1^2$, so

$$\begin{gathered} x_1(r - x_3) - x_2 = x_1(r - \frac{1}{b}x_1^2 - 1) = 0 \\ x_1 = 0 \text{ or } x_1 = \pm\sqrt{b(r-1)} \end{gathered}$$

giving three critical points $(0, 0, 0)$, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$.

The origin is a critical point for all parameter values, and the derivative of f there is

$$Df(\mathbf{0}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{5}$$

with eigenvalues $-b$ and $\frac{1}{2}\left[-(\sigma+1) \pm \sqrt{\sigma^2 + (4r-2)\sigma + 1}\right]$.

If $r < 1$ then $4r - 2 < 2$ and $\sigma^2 + (4r-2)\sigma + 1 < (\sigma+1)^2$, so all three eigenvalues have negative real part and the origin is a sink. In fact, when $r < 1$ the origin is the only critical point and it is a globally stable equilibrium point.

When $r > 1$, a pitchfork bifurcation occurs and two new critical points appear. Lorenz called these points

$$\begin{aligned} C^+ &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \text{ and} \\ C^- &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1). \end{aligned}$$

And if $r > 1$ then $\sigma^2 + (4r-2)\sigma + 1 > (\sigma+1)^2$, so two eigenvalues of $Df(\mathbf{0})$ have negative real part and one has positive real part. Therefore the origin becomes a saddle, with a one-dimensional unstable manifold and a two-dimensional stable manifold. This stable manifold separates the basins of C^+ and C^-.(add citation, Global invariant manifolds in the transition to preturbulence in the Lorenz system)

Following Strogatz 2015 we can also determine the stability of the critical points C^+ and C^- under certain conditions.

The derivative of f at C^+ is

$$Df(C^+) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix} \tag{6}$$

The derivative of f at C^- is

$$Df(C^+) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{bmatrix} \tag{7}$$

Both have the same characteristic equation:

$$\lambda^3 + [b + \sigma + 1]\lambda^2 + [br + b\sigma]\lambda + 2br\sigma - 2b\sigma = 0$$

This is intractable generally, but when $r = r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ this characteristic equation becomes

$$\begin{aligned} 0 &= \lambda^3 + [b + \sigma + 1]\lambda^2 + \left[b\left(\sigma\frac{\sigma + b + 3}{\sigma - b - 1}\right) + b\sigma\right]\lambda + 2b\left(\sigma\frac{\sigma + b + 3}{\sigma - b - 1}\right)\sigma - 2b\sigma \\ &= \lambda^3 + [b + \sigma + 1]\lambda^2 + \left[\frac{\sigma + b + 3}{\sigma - b - 1} + 1\right]b\sigma\lambda + 2b\sigma\left[\sigma\frac{\sigma + b + 3}{\sigma - b - 1} - 1\right] \\ &= \lambda^3 + [b + \sigma + 1]\lambda^2 + \left[\frac{2\sigma + 2}{\sigma - b - 1}\right]b\sigma\lambda + 2b\sigma^2\left[\frac{\sigma + b + 3}{\sigma - b - 1} - \frac{1}{\sigma}\right] \end{aligned}$$

with the eigenvalues

$$\lambda_1 = -\sigma - b - 1, \quad \lambda_2 = \frac{\sqrt{2b\sigma^2 + 2b\sigma}}{\sqrt{b + 1 - \sigma}}, \quad \lambda_3 = -\frac{\sqrt{2b\sigma^2 + 2b\sigma}}{\sqrt{b + 1 - \sigma}}.$$

So when $r = r_H$ and $\sigma > b + 1$, the second and third eigenvalues are purely imaginary, and a Hopf bifurcation occurs (thus the label r_H). When $1 < r < r_H$, all three eigenvalues are negative and C^+ and C^- are stable equilibrium points.

Interestingly, Strogatz claims that when $r > r_H$, C^+ and C^- are encircled by *saddle cycles*, with a two-dimensional stable manifold and a two-dimensional unstable manifold (sic) (Strogatz 2015). Unfortunately I am not yet able to fully understand or evaluate this surprising claim.

Visualization

The following plots show how the trajectories from 8 initial values at the corners of a cube $(\pm 1, \pm 1, \pm 1)$ change as r increases, for t from 0 to 100, with $\sigma = 10$ and $b = 8/3$.

When $r < 1$, all trajectories approach the origin, the only critical point. When r increases past 1, two new critical points appear and these trajectories now approach these new critical points, which are initially stable. As r increases, these critical points move further from the origin and spiral in more slowly, gradually forming the "butterfly wings".

Unfortunately the Hopf bifurcation at $r = r_H \approx 24.7$ as these critical points become unstable is not visible. In fact, numerical simulation with MATLAB showed that even when these points are stable equilibria ($r < r_H$), convergence to these points takes longer and longer as r approaches r_H, and for r near r_H convergence does not occur until long after $t = 100$.

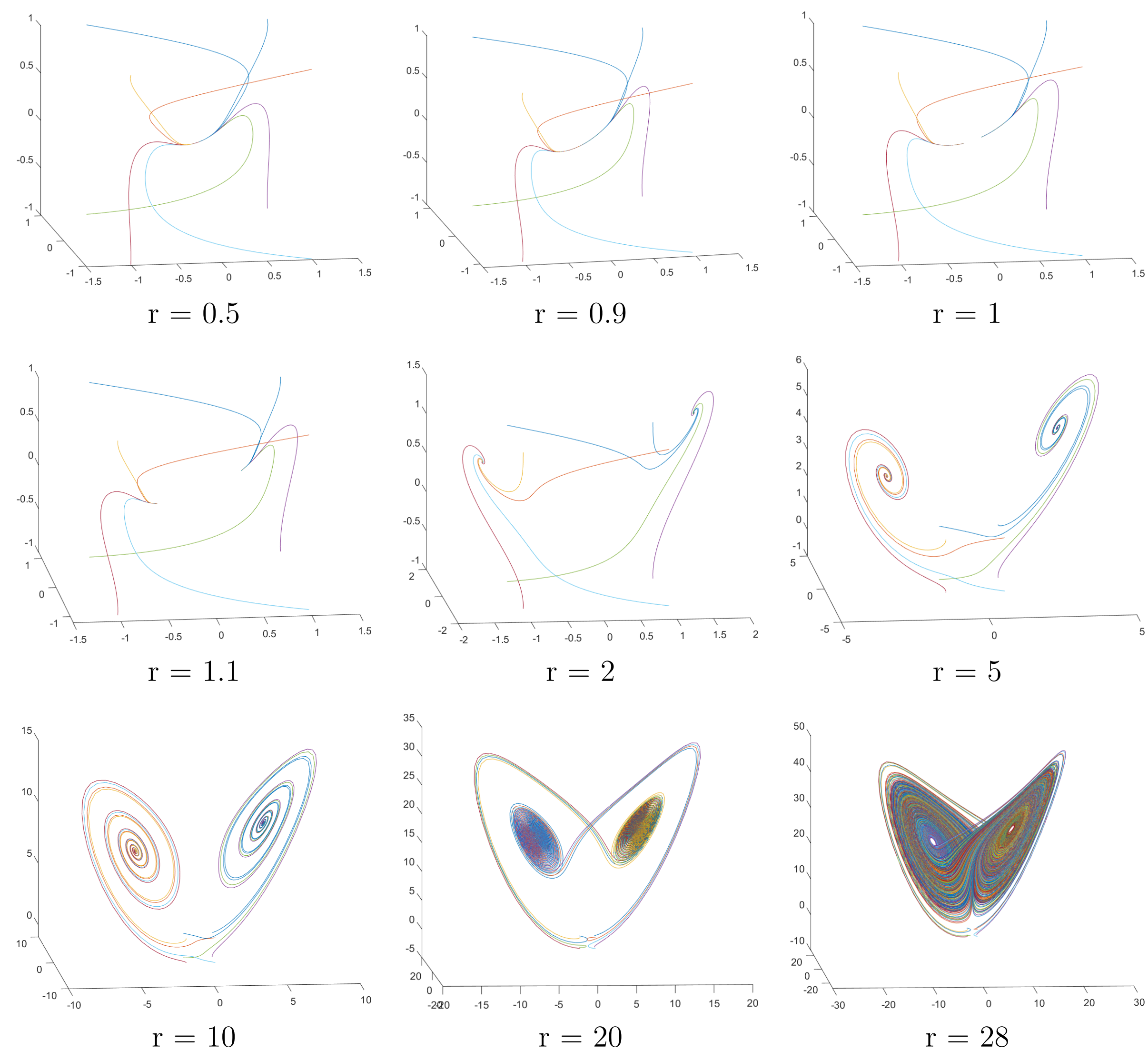


Figure 2: Progression of the Lorenz system as r increases

Chaotic Behavior

By far the most interesting behavior of the Lorenz system occurs after the Hopf bifurcation, and although this was not a course in chaos theory, this deserves at least a brief discussion.

In particular, Lorenz studied the parameter values $\sigma = 10, r = 28, b = 8/3$ and discovered chaotic behavior, that is, sensitive dependence on initial conditions. Lorenz famously described this as

> Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

For example, Table 1 shows the evolution of three nearby points under the Lorenz system. Although the points initially follow similar trajectories, they soon diverge into very different results.

t	x_1	x_2	x_3	x_1	x_2	x3	x_1	x_2	x_3
0	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0001	1.0000	1.0000
10	-4.9189	-3.8105	24.6302	-4.9188	-3.8105	24.6301	-4.9188	-3.8105	24.6300
20	0.4654	2.3800	22.0635	0.4662	2.3894	22.0848	0.4674	2.3984	22.1038
30	11.2932	9.9830	32.0380	8.9111	16.5646	12.4624	12.7990	17.6631	26.6436
40	12.8148	6.6362	38.3562	1.4247	0.7080	20.4808	10.5001	1.1823	37.9576
50	1.9468	-2.2861	26.9283	-1.3758	-2.3948	15.8657	7.5134	0.5774	33.2759
60	1.4028	-0.6762	23.4386	2.6514	4.7171	11.5649	-1.0160	2.3426	25.3731
70	7.0489	1.2019	31.9182	-17.0124	-17.4941	38.5163	-1.6267	-1.4362	19.2221
80	1.8661	-1.4101	25.6092	-4.2912	-4.1732	22.3125	4.6893	7.4489	15.6602
90	3.1975	5.1125	20.1064	-6.1280	-11.2512	11.3329	3.3108	5.5928	13.9516
100	6.0875	2.3285	29.0413	8.1564	10.8939	22.3339	-2.2546	-5.6657	25.1917

Table 1: Evolution of the Lorenz system from three close initial values

Lorenz Attractor

Although these values appear random, it's obvious from plots of the Lorenz system that many trajectories are eventually confined to a small region of space that resembles a butterfly. This region is the Lorenz attractor.

An attractor is a subset A of the phase space that is forward invariant under f (if a is an element of A then so is f(t,a), for all $t > 0$) and for which there exists a basin of attraction, B(A), such that for any $b \in B(A)$ and any open neighborhood N of A, there is a positive constant T such that $f(t, b) \in N$ for all real $t > T$.

In fact, for the Lorenz system, almost all trajectories (any trajectory from an initial point not on the two-dimensional stable manifold of the origin), will approach the Lorenz attractor. In other words, the basin of attraction of the Lorenz attractor is all of $\mathbb{R}^3$ except the two-dimensional stable manifold at the origin.

In 1976, John Guckenheimer developed a geometric model with properties similar to the Lorenz system (Guckenheimer 1976), and in 2002, Warwick Tucker proved that this model was indeed equivalent to the Lorenz system (Tucker 2002). Among other things this proved that the Lorenz attractor was a fractal (and therefore a strange attractor).

Summary

To summarize: for any parameters and any initial condition, a solution of the Lorenz system exists near $t = 0$ and is bounded as t increases, and therefore exists for all $t > 0$.

For $r < 1$, the origin is only equilibrium point and is stable. All trajectories go to the origin as $t \to \infty$.

When $r > 1$, the origin becomes a saddle with a two-dimensional stable manifold, and two new equilibrium points appear:

$$C^+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \text{ and}$$

$$C^- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

For $r < r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1)$, these equilibrium points are stable.

For $r > r_H$, and in particular for the parameters Lorenz studied, $\sigma = 10, r = 28, b = 8/3$, trajectories from any initial point outside the two dimensional stable manifold of the origin approach but never reach the critical points C^+ and C^-. Some solutions follow chaotic trajectories near the Lorenz attractor.

This chaotic behavior is the reason the Lorenz system is famous and interesting, but unfortunately, this is the subject of another course in chaos theory and could only be briefly mentioned in this paper. However, the results in this paper are an essential starting point for studying the chaotic behavior of the Lorenz system.

Appendix: MATLAB code

To make the plots in Figure 1. Viewing angles were chosen and plots were exported to PNG files manually.

```
sigma = 10;
b = 8/3;
r = 28;
f = @(t,x) ...
      [ ...
       sigma*(x(2)-x(1)); ...
       x(1)*(r - x(3)) - x(2); ...
       x(1)*x(2) - b * x(3)   ...
      ];
[t,x] = ode45(f,[0 100],[1 1 1]);
plot3(x(:,1),x(:,2),x(:,3))
```

To make the plots in Figure 2, showing the changes in the Lorenz system as r increases from 0.5 to 28, starting from the eight corners of the -1 to +1 cube. Interesting values of r were chosen and changed in this script, and plots were exported to PNG files manually.

```
sigma = 10;
b = 8/3;
r = 28;
f = @(t,x) ...
      [ ...
       sigma*(x(2)-x(1)); ...
       x(1)*(r - x(3)) - x(2); ...
       x(1)*x(2) - b * x(3)   ...
      ];
for i = 1:8
    [t,x] = ode45(f,[0 100], ...
        [(-1)^i (-1)^floor(i/2) (-1)^floor(i/4)]);
    plot3(x(:,1),x(:,2),x(:,3))
    hold on
end
hold off
```

To make Table 1.

```
sigma = 10;
b = 8/3;
r = 28;
f = @(t,x) ...
      [ ...
       sigma*(x(2)-x(1)); ...
       x(1)*(r - x(3)) - x(2); ...
       x(1)*x(2) - b * x(3)   ...
      ];

M1 = zeros(11, 4);
M2 = zeros(11, 4);
M3 = zeros(11, 4);

M1(1,2:4) = [0.9999 1 1];
M2(1,2:4) = [1.0000 1 1];
M3(1,2:4) = [1.0001 1 1];

for i = 1:10
     t = i*10;
     [t1,x1] = ode45(f,[0 t], M1(1, 2:4));
     M1(i+1, 1) = t;
     M1(i+1, 2) = x1(end, 1);
     M1(i+1, 3) = x1(end, 2);
     M1(i+1, 4) = x1(end, 3);

     [t2,x2] = ode45(f,[0 t], M2(1, 2:4));
     M2(i+1, 1) = t;
     M2(i+1, 2) = x2(end, 1);
     M2(i+1, 3) = x2(end, 2);
     M2(i+1, 4) = x2(end, 3);

     [t3,x3] = ode45(f,[0 t], M3(1, 2:4));
     M3(i+1, 1) = t;
     M3(i+1, 2) = x3(end, 1);
     M3(i+1, 3) = x3(end, 2);
     M3(i+1, 4) = x3(end, 3);
end

for i = 1:11
     fprintf(['%d & %0.4f & %0.4f & %0.4f & %0.4f & %0.4f & ' ...
              '%0.4f & %0.4f & %0.4f & %0.4f \\\\ \n'], ...
         M1(i, 1), ...
         M1(i, 2), M1(i, 3), M1(i, 4), ...
         M2(i, 2), M2(i, 3), M2(i, 4), ...
         M3(i, 2), M3(i, 3), M3(i, 4) ...
     )

end
```

References

Guckenheimer, John (1976). “A Strange, Strange Attractor”. In: *The Hopf Bifurcation and Its Applications*. Springer, pp. 368–381.

Lorenz, Edward N. (Mar. 1, 1963). “Deterministic Nonperiodic Flow”. In: *Journal of the Atmospheric Sciences* 20.2, pp. 130–141. ISSN: 0022-4928. DOI: 10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2.

Perko, Lawrence (2001). *Differential Equations and Dynamical Systems*. 3rd ed. Texts in Applied Mathematics. New York: Springer-Verlag. ISBN: 978-0-387-95116-4. DOI: 10.1007/978-1-4613-0003-8.

Strogatz, Steven H. (Mar. 1, 2015). *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering, Second Edition*. 2 edition. Boulder, CO: Westview Press. 531 pp. ISBN: 978-0-8133-4910-7.

Tucker, Warwick (2002). “A Rigorous ODE Solver and Smale’s 14th Problem”. In: *Foundations of Computational Mathematics* 2.1, pp. 53–117.