MAT 272 Test 3 and Final Exam Review

13.1 Double Integrals over Rectangular Regions

1. Compute a Riemann sum approximation of

$$\iint_D f(x,y)dA$$

where $D = [-1,1]^2$ (the square of all points (x,y) with $-1 \le x \le 1, -1 \le y \le 1$), based on the following information:
$f\left(-\frac{1}{2},-\frac{1}{2}\right) = 1, f\left(\frac{1}{2},\frac{1}{2}\right) = 2, f\left(\frac{1}{2},-\frac{1}{2}\right) = 3, f\left(-\frac{1}{2},\frac{1}{2}\right) = 4.$
   a. 10
   b. 2.5
   c. 40
   d. 20
   e. None of the above.

13.2 Double Integrals over General Regions

1. Calculate $\iint_R 2xydA$ where R is the region between the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$.

2. Reverse the order of integration on the following double integral.

$$\int_1^4 \int_1^{\sqrt{x}} f(x,y)dy\,dx$$

   a. $\int_1^{\sqrt{y}} \int_1^4 f(x,y)dx\,dy$
   b. $\int_1^2 \int_0^{y^2} f(x,y)dx\,dy$
   c. $\int_1^{\sqrt{y}} \int_1^2 f(y,x)dx\,dy$
   d. $\int_1^2 \int_{y^2}^4 f(x,y)dx\,dy$
   e. None of the above.

3. Evaluate the integral. Show all your steps.

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$$

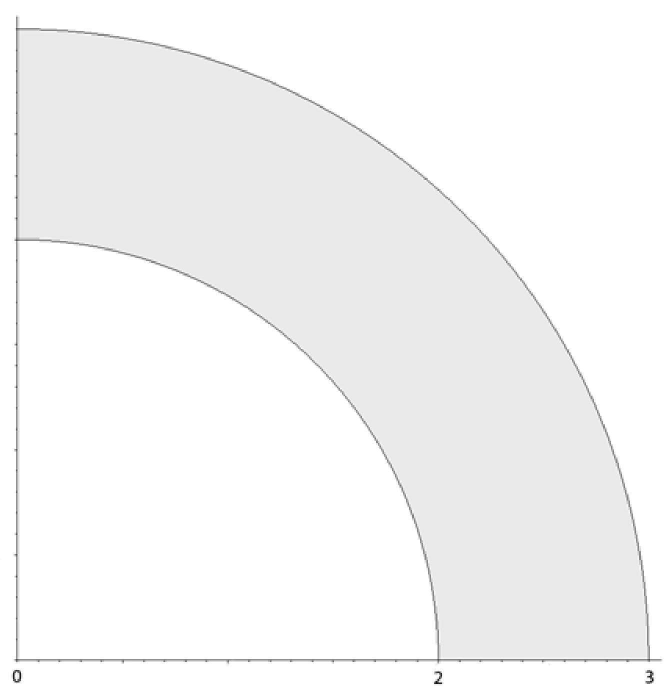
13.3 Double Integrals in Polar Coordinates

1. Evaluate the iterated integral by converting to polar coordinates:

$$\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dydx$$

   a. $\frac{\pi}{2}(e^4 - 1)$
   b. πe^4
   c. $\frac{\pi}{4}e^4$
   d. $2\pi(e^4 - 1)$
   e. None of the above.

2. Find the *exact* volume under $z = \sqrt{x^2 + y^2}$ over the region in the *xy* plane given below.



3. In an ancient city, the royal palace grounds occupied the center disk of radius 1 kilometer. Common people lived in a ring-shaped region with inner radius 1 kilometer and outer radius 3 kilometers that was approximately uniformly filled with residences. What was the average distance of a common residence in that city from the city center?

13.4 Triple Integrals

1. Set up (do not evaluate) a triple integral that represents the volume in the first octant below the plane $2x + 3y + z = 6$. Use the integration order dxdydz.

13.5 Triple Integrals in Cylindrical and Spherical Coordinates

1. Evaluate

$$\iiint_V (x^2 + y^2)\, dV$$

where V is the volume bounded between the plane $z = 0$ and the paraboloid $z = 16 - x^2 - y^2$.

2. Express the volume of the solid inside the sphere $\rho = 2$ and outside the cylinder $x^2 + y^2 = 1$ using a single triple integral in spherical coordinates. Sketch the volume in a 2d coordinate system that shows the xy-plane as the first axis and the z axis as the second axis. You do not have to evaluate, but you have to show your work in determining limits of integration.

13.6 Integrals for Mass Calculations

1. Find the centroid of the region in $\mathbb{R}^2$ that is bounded by the curve $y = \sqrt[3]{x}$, the x-axis and the line $x = 8$.

2. A spherical shell with inner radius 2 meters and outer radius 3 meters is filled with gas. The mass density function of the gas (in kilogram per cubic

meters) is $\varrho(\rho) = \frac{1}{\rho}$ where ρ is the distance from the center, in meters. Find the total mass of the gas in kilograms.

a. 5π
b. 10π
c. 20π
d. $\frac{76\pi}{3}$
e. None of the above.

3. A town is the shape of a rectangle, with vertices (0,0), (8,0), (8,4) and (0,4). The population density is modeled by the function $(x, y) = xy^2$. They want to place their city hall so that it is centered relative to their population, not necessarily geographically. Determine the "population center" of this town. (Assume the units are miles, and the density is "hundreds of people per square mile").

13.7 Change of Variables in Multiple Integrals

1. Let (u, v, w) be a new coordinate system for $\mathbb{R}^3$ defined by

$$x = 2u\cos v\,,$$
$$y = 3u\sin v\,,$$
$$z = 5w$$

with $u \geq 0, 0 \leq v < 2\pi$ and w an arbitrary real number.

Evaluate the Jacobian $J(u, v, w) = \frac{\partial(x,y,z)}{\partial(u,v,w)}$.

a. $\frac{30u\cos v\sin v}{v}$
b. $30u$
c. $10u$
d. $10v$
e. None of the above.

2. Evaluate the double integral

$$\iint_D (x+y)^2(y-x)^4 dA$$

over the region D using a change to a new coordinate system. D is the square with vertices (1,0), (0,1), (-1,0) and (0,-1).

3. Let the polar coordinate system be changed (for this problem only) by the introduction of a factor 2 into the x coordinate, $x = 2r\cos\theta,\ y = r\sin\theta$, so that the curves r=constant are no longer circles, but ellipses, and call these coordinates (r,θ) elliptic polar coordinates.
   a. Compute the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$.
   b. Use the Jacobian from a. to set up and evaluate a double integral in elliptic polar coordinates that represents the area of the ellipse with major radius 2 and minor radius 1.

14.1 Vector Fields

1. Find the gradient field corresponding to $\varphi(x,y,z) = \sqrt{y^2 - x^2 + xe^{z^2} - y^3 z}$.

2. Find a potential φ for the vector field $F(x,y) = < 6x\sin y, 3\cos x >$, if it exists.
   a. $\varphi(x,y) = < 3x^2 \sin y, 3\cos x\, y >$
   b. $\varphi(x,y) = 18x\cos x \sin y$
   c. $\varphi(x,y) = 3x^2 \sin y + C(y)$
   d. $\varphi(x,y) = 6\sin y$
   e. No potential exists.

14.2 Line Integrals

1. Calculate the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F} = < 2x, y^2 >$ and C is the straight line path from (1,-2) to (2,-1).

2. Evaluate the line integral of $\boldsymbol{F}(x,y) = < -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} >$ over the semicircle C centered at $(0,0)$ from $(1,0)$ to $(-1,0)$.

14.3 Conservative Vector Fields

1. Which one of the following vector fields is <u>not</u> conservative?

   a. $F(x,y) =< ye^{xy}, xe^{xy} >$
   b. $F(x,y) =< 6x^2y, 2x^3 - 2y >$
   c. $F(x,y) =< 1{,}1>$
   d. $F(x,y) =< y, x >$
   e. All of the above are conservative.

2. Evaluate the line integral (work) of the vector field

$$F(x,y) =< 3x^2y + e^y, x^3 + xe^y >$$

   over the curve C that consists of the line segments from (1,0) to (1,1), from (1,1) to (-1,1) and from (-1,1) to (-1,0).

3. What is the meaning of path independence of a vector field F?

   a. All line integrals of F (over any path) have the same value.

   b. All line integrals of F between two fixed points have the same value.

   c. The vector field assumes the same values on any two paths between two fixed points.

   d. No matter which path you take from point P to point Q, F(Q) is always going to be the same value.

   e. F(Q)-F(P) has the same value for all points P and Q in the domain of the vector field.

14.4 Green's Theorem

1. Evaluate $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the circle with radius 2 and $\boldsymbol{F}(x,y) =< 3x^2y + 2x - 4y, x^3 + 5x + e^{y^2} >$.
   a. 36π
   b. 48π

c. 4π

d. 18π

e. None of the above.

2. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F} =< 2x + 4y, 3x - y >$ and C is the path from the origin to the point (4,0) to the point (3,10) and back down to the origin.

3. Suppose C is the following curve: a straight line from $(0,0)$ to $(2,0)$, a straight line from $(2,0)$ to $(2,1)$ and a half-circle with center $(1,1)$ from $(2,1)$ to $(0,1)$. Observe that this is **<u>not</u>** a closed curve. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ with $\boldsymbol{F}(x, y) =< -y, x >$.

14.5 Divergence and Curl

1. If $\boldsymbol{F}$ is a sufficiently differentiable vector field in $\mathbb{R}^3$, which one of the following objects is defined and a scalar function?

   a. $\nabla \times (\nabla \times \boldsymbol{F})$

   b. $\nabla \cdot (\nabla \times \boldsymbol{F})$

   c. $\nabla(\nabla \cdot \boldsymbol{F})$

   d. $\nabla \cdot (\nabla \cdot \boldsymbol{F})$

   e. They are all either undefined or vector fields.

14.6 Surface Integrals

1. Which one of the following vectors is the outward unit normal of the unit sphere at $(-1,0,0)$?

   a. $< 1,0,0 >$

   b. $< -1,0,0 >$

   c. $< 0,0,1 >$

   d. $< 0,0,-1 >$

2. For the parametric surface $\boldsymbol{r}(u, v) = < u\cos(v), u\sin(v), 0 >$ with $0 \le u \le 1$ and $0 \le v < 2\pi$, find the (vectorial) surface element $d\boldsymbol{S}$.

   a. $< 0,0,1 > u\, du\, dv$
   b. $< 0,0,1 > v\, du\, dv$
   c. $< 0,0,1 >\ du\, dv$
   d. $< 1,1,0 > du\, dv$

3. Evaluate the flux integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{N} dS$ for $\boldsymbol{F} = < y, z, x >$ and S is that portion of the plane $z = 2 - x - y$ above the square $0 \le x \le 1, 0 \le y \le 1$ with upward normal.

14.7 Stokes' Theorem

1. Let S be the oriented surface that is the upper half unit sphere (the set of points with $x^2 + y^2 + z^2 = 1, z \ge 0$) with the upward normal. Let $\boldsymbol{F}(x, y, z) = < z, x, y >$.

   a. Evaluate the curl of $\boldsymbol{F}$.

   b. Evaluate

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$$

2. Let a vector field $\boldsymbol{F}$ be defined for all points in $\mathbb{R}^3$ except the origin by

$$\boldsymbol{F}(x, y, z) = \frac{< x, y, z >}{x^2 + y^2 + z^2}$$

   Demonstrate explicitly that the curl of F is zero. Does this guarantee that $\boldsymbol{F}$ is conservative? Explain.

14.8 Divergence Theorem

1. Evaluate the outward flux of $\boldsymbol{F}(x, y, z) =< x + y, y + z, z + x >$ across the surface of the cube $[-1,1]^3$.
    a. 1
    b. 3
    c. 8
    d. 24
    e. None of the above.
2. Determine the flux of $\boldsymbol{F}(x, y, z) =< 3x + 2y, x - y, 2x + y + z >$ through the four-sided object in the first octant bounded by the plane $x + 2y + 3z = 6$ and the xy, xz and yz planes.
3. Let S be the oriented surface that is the upper half unit sphere (the set of points with $x^2 + y^2 + z^2 = 1, z \geq 0$) with the upward normal. Observe that this is an <u>open</u> surface – the unit disk in the xy plane is not part of it. Let $\boldsymbol{F}(x, y, z) =< x + 2y + 3z, 5e^x, x^2 + y^2 + z >$ be a vector field.

    a. Evaluate the divergence of $\boldsymbol{F}$.
    b. Evaluate the flux of $\boldsymbol{F}$ through S.

Answers

13.1 Double Integrals over Rectangular Regions

1. A

13.2 Double Integrals over General Regions

1. $\iint_R 2xydA = \int_0^9 \int_{\frac{1}{3}x}^{\sqrt{x}} 2xy\, dydx = \int_0^9 x\left(x - \frac{1}{9}x^2\right) dx = \left[\frac{1}{3}x^3 - \frac{1}{36}x^4\right]_0^9 = \frac{243}{4}$
2. D
3. On this integral we must change the order of integration since e^{x^3} has no closed-form antiderivative. The integration region is bounded by the curves x=1, y=0 and the parabola $y = x^2$.

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx\, dy = \int_0^1 \int_0^{x^2} e^{x^3} dy\, dx = \int_0^1 e^{x^3} x^2 dx = \frac{1}{3} e^{x^3}\Big|_0^1 = \frac{1}{3}(e-1)$$

13.3 Double Integrals in Polar Coordinates

1. A
2. $V = \int_0^{\frac{\pi}{2}} \int_2^3 r^2 dr d\theta = \int_0^{\frac{\pi}{2}} d\theta \cdot \int_2^3 r^2 dr = \frac{\pi}{2} \cdot \frac{1}{3} r^3 \Big|_2^3 = \frac{19\pi}{6}$
3. We use the definition of average of a function of two variables – integral over the integration region divided by the area of the region in polar coordinates:

$$\bar{r} = \frac{\int_0^{2\pi} \int_1^3 r^2 dr d\theta}{\pi(3^2 - 1^2)} = \frac{2\pi \cdot \frac{1}{3}(3^3 - 1^3)}{\pi(3^2 - 1^2)} = \frac{13}{6}$$

Observe that the average distance is greater than the average of the radii of the annulus. Think about why this has to be the case.

13.4 Triple Integrals

1. The integral is

$$\int_0^6 \int_0^{(2-\frac{1}{3}z)} \int_0^{(3-\frac{3}{2}y-\frac{1}{2}z)} dxdydz$$

13.5 Triple Integrals in Cylindrical and Spherical Coordinates

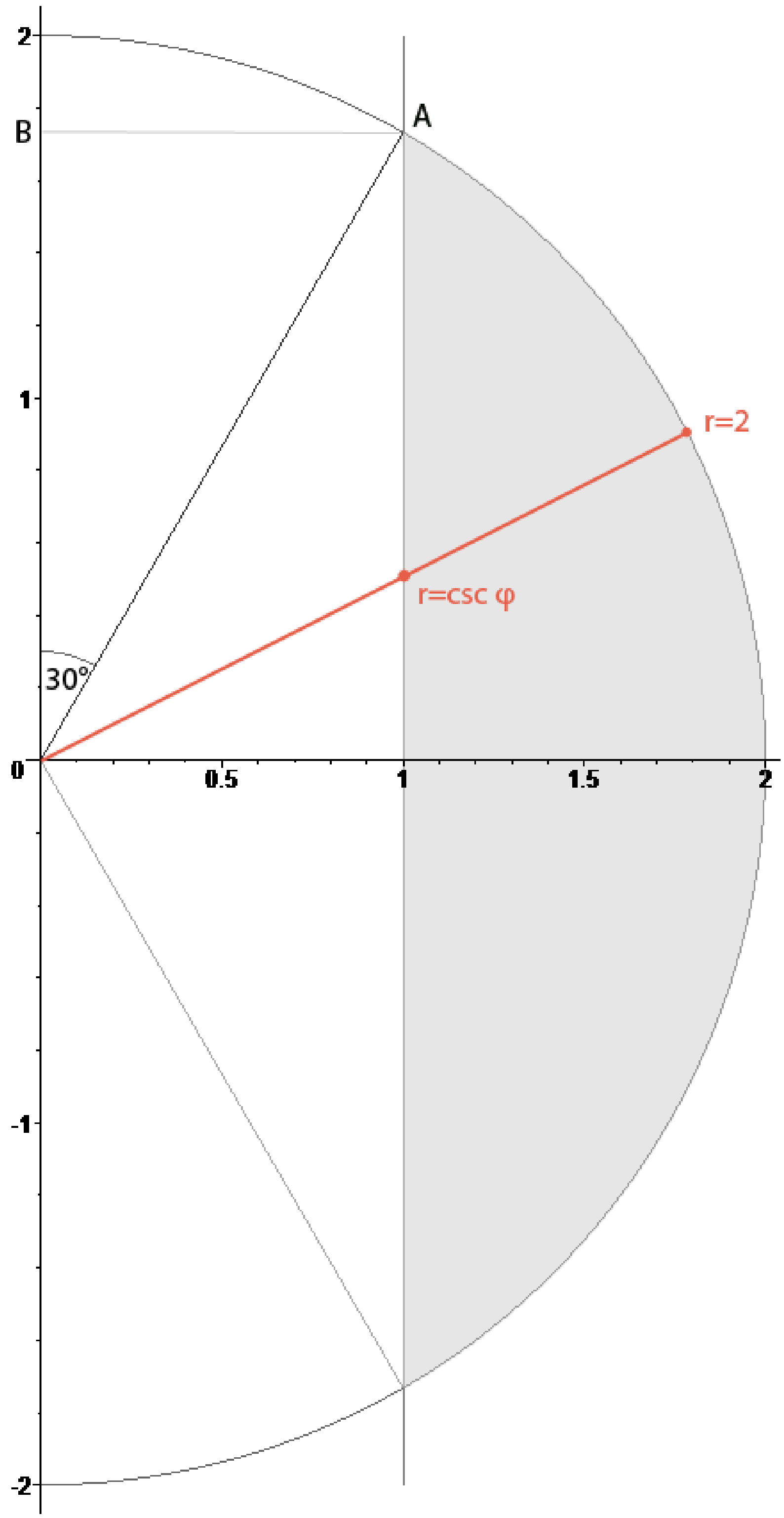
1. We evaluate this integral in cylindrical coordinates:

$$\iiint_V (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} r^2\, dz\, rdr\, d\theta$$

$$= \int_0^{2\pi} \int_0^4 r^2\,(16 - r^2) rdr\, d\theta = 2\pi \int_0^4 (16r^3 - r^5) dr$$

$$= 2\pi\,(4r^4 - \frac{1}{6} r^6)\Big|_0^4 = 2\pi(4^5 - \frac{1}{6}4^6) = \frac{2048\pi}{3}$$

2. The sketch of the situation could look like this:



The gray shaded area represents the volume. Its polar angle (φ value) ranges from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$. The angle $\frac{\pi}{6}$ is determined as $\cos^{-1}\left(\frac{1}{2}\right)$ using the right triangle OAB. θ ranges from 0 to 2π due to the rotational symmetry of the volume with respect to the z axis.

Limits for ρ depend on the φ value. The red line represents a fixed φ value. The lower limit for ρ is the hypotenuse of a right triangle with angle φ opposite to a side of length 1. Therefore, $\sin\varphi = \frac{1}{\rho}$ or $\rho = \csc\varphi$. The upper limit for ρ is 2. Therefore, the desired integral is

$$V = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc\varphi}^{2} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

13.6 Integrals for Mass Calculations

1. We first compute the moments:

$$M_y = \int_0^8 \int_0^{\sqrt[3]{x}} x dy dx = \int_0^8 x^{\frac{4}{3}} dx = \frac{3}{7} 8^{\frac{7}{3}} = \frac{384}{7}$$

$$M_x = \int_0^8 \int_0^{\sqrt[3]{x}} y dy dx = \frac{1}{2} \int_0^8 x^{\frac{2}{3}} dx = \frac{1}{2} \cdot \frac{3}{5} 8^{\frac{5}{3}} = \frac{48}{5}$$

Next we compute the “mass” (area) of the region:

$$M = \int_0^8 \int_0^{\sqrt[3]{x}} dy dx = \int_0^8 x^{\frac{1}{3}} dx = \frac{3}{4} 8^{\frac{4}{3}} = 12$$

Therefore, the centroid is $\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{32}{7}, \frac{4}{5}\right)$.

2. B
3. We first compute the moments:

$$M_y = \int_0^4 \int_0^8 x^2 y^2 dx dy = \int_0^4 y^2 dy \int_0^8 x^2 dx = \frac{32768}{9}$$

$$M_x = \int_0^4 \int_0^8 x y^3 dx dy = \int_0^4 y^3 dy \int_0^8 x dx = 2048$$

The mass is

$$M = \int_0^4 \int_0^8 x y^2 dx dy = \int_0^4 y^2 dy \int_0^8 x dx = \frac{2048}{3}$$

From that, we get the coordinates of the center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{16}{3}$$

$$\bar{y} = \frac{M_x}{M} = 3$$

13.7 Change of Variables in Multiple Integrals

1. B
2. We introduce new coordinates: $u = x + y$ and $v = y - x$. In these new coordinates, D is the square $[-1,1]^2$. To save the labor of having to invert this variable transformation, we'll evaluate the Jacobian of the inverse transformation:

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

It follows that

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{2}$$

Therefore,

$$\iint_D (x+y)^2(y-x)^4 dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u^2v^4 dudv$$

We simplify by exploiting symmetry:

$$\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u^2v^4 dudv = 2\int_{0}^{1}\int_{0}^{1} u^2v^4 dudv = 2\cdot\frac{1}{3}\cdot\frac{1}{5} = \frac{2}{15}$$

3. For the given coordinate transformation, the Jacobian is $\frac{\partial(x,y)}{\partial(r,\theta)} = 2rdrd\theta$. In the new coordinates, the ellipse is the unit circle. Therefore, the area is

$$\int_{0}^{2\pi}\int_{0}^{1} 2rdrd\,\theta = 2\pi$$

14.1 Vector Fields

1. By taking partial derivatives, we find

$$\nabla\varphi = \frac{1}{2\sqrt{y^2 - x^2 + xe^{z^2} - y^3 z}} < -2x + e^{z^2}, 2y - 3y^2 z, 2xze^{z^2} - y^3 >$$

2. E.

14.2 Line Integrals

1. A parametrization of the indicated path is $\boldsymbol{r}(t) =< 1,-2 > +t < 1,1 >$, $0 \le t \le 1$. Then $d\boldsymbol{r} =< 1,1 > dt$. Thus

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 (2(1+t)\cdot 1 + (-2+t)^2\cdot 1)dt = \int_0^1 (6 - 2t + t^2)dt$$

$$= 6t - t^2 + \frac{1}{3}t^3\Big|_0^1 = 6 - 1 + \frac{1}{3} = 5\frac{1}{3}$$

2. A parametrization of the semicircle is $\boldsymbol{r}(t) =< \cos t, \sin t >$, $0 \le t \le \pi$. Then $d\boldsymbol{r} =< -\sin t, \cos t > dt$ and $\boldsymbol{F}(\boldsymbol{r}(t)) =< -\sin t, \cos t >$ and therefore

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\mathbf{0}}^{\boldsymbol{\pi}} 1dt = \pi$$

14.3 Conservative Vector Fields

1. E
2. Since $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} = 3x^2 + e^y$ and the domain of F is $\mathbb{R}^2$, which is simply connected, F is conservative. A potential for F is $\varphi = x^3y + xe^y$. By using the fundamental theorem for line integrals, we evaluate the line integral as

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \varphi(-1,0) - \varphi(1,0) = -1 - 1 = -2$$

3. B

14.4 Green's Theorem

1. A
2. The 2d curl of $\boldsymbol{F}$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 3 - 4 = -1$. By Green's theorem,

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_T -dA = -\text{area(T)} = -20$$

where T is the triangle with the indicated vertices. The area of that triangle is $\frac{1}{2}$base $\cdot$ height $=$20.

3. Suppose C is the following curve: a straight line from (0,0) to (2,0), a straight line from (2,0) to (2,1) and a half-circle with center (1,1) from (2,1) to (0,1). Observe that this is <u>not</u> a closed curve. Find the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y) =< -y, x >$.

By adding the line segment L from (0,1) to the origin, we close the path. Let's call this closed path D and the region enclosed by it R. Green's theorem then applies and yields

$$\int_D \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 2dA = 2\cdot \text{area(R)}$$

R consists of a rectangle of area 2 and a half-circle of radius 1. The area of R is therefore $2+\frac{\pi}{2}$. It follows that

$$\int_D \boldsymbol{F}\cdot d\boldsymbol{r} = 4+\pi$$

The line integral of $\boldsymbol{F}$ over D is the line integral of $\boldsymbol{F}$ over C, plus the line integral of $\boldsymbol{F}$ over L:

$$\int_D \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} + \int_L \boldsymbol{F}\cdot d\boldsymbol{r}$$

But $\boldsymbol{F}(x,y) =< -y, 0 >$ on the line segment L, which is orthogonal to the direction of L. Therefore $\boldsymbol{F}\cdot d\boldsymbol{r} = \boldsymbol{0}$ on L and $\int_L \boldsymbol{F}\cdot d\boldsymbol{r} = 0$. Hence

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_D \boldsymbol{F}\cdot d\boldsymbol{r} = 4+\pi$$

14.5 Divergence and Curl

1. B

14.6 Surface Integrals

1. B
2. A
3. By using x and y as parameters, we obtain a parametrization of the surface S:

$$\boldsymbol{r}(u,v) =< u, v, 2-u-v >$$

Next we compute the partial derivatives:

$$\boldsymbol{r}_u\,(u,v) =< 1,0,-1 >$$
$$\boldsymbol{r}_v\,(u,v) =< 0,1,-1 >$$

To find a surface normal, we take the cross product:

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} =< 1,1,1 >$$

This is indeed the upward normal. We recall $\boldsymbol{n}\, dS = (\boldsymbol{r}_u \times \boldsymbol{r}_v)\, du\, dv$ and find

$$\boldsymbol{n}\, dS =< 1,1,1 > du\, dv$$

Furthermore, $\boldsymbol{F}(u, v) =< v, 2 - u - v, u >$, thus

$$\boldsymbol{F} \cdot \boldsymbol{n}\, dS = v + 2 - u - v + u = 2$$

Therefore, $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \int_0^1 \int_0^1 2\, du\, dv = 2.$

14.7 Stokes' Theorem

1. Let S be the oriented surface that is the upper half unit sphere (the set of points with $x^2 + y^2 + z^2 = 1, z \geq 0$) with the upward normal. Let $\boldsymbol{F}\,(x, y, z) =< z, x, y >$.
   a. $\nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{vmatrix} =< 1,1,1 >$
   b. It is a consequence of Stokes' theorem that the flux of $\nabla \times \boldsymbol{F}$ through S is the same as the flux of $\nabla \times \boldsymbol{F}$ through the upward oriented unit disk D in the xy plane, since they share the same boundary curve. Hence

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_D < 1,1,1 > \cdot\, d\boldsymbol{S}$$

   D has the normal $\boldsymbol{n} =< 0,0,1 >$, thus $< 1,1,1 > \cdot\, d\boldsymbol{S} = dS$. It follows that

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_D dS = \text{area of D} = \pi$$

2.

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$$

We show only that the first component of the curl is zero since the calculation is the same for all three components except for renaming of variables.

$$\partial_y \frac{z}{x^2+y^2+z^2} - \partial_z \frac{y}{x^2+y^2+z^2} = \frac{-2yz}{(x^2+y^2+z^2)^2} + \frac{2yz}{(x^2+y^2+z^2)^2} = 0$$

The domain of $\boldsymbol{F}$ is the "punctured" $\mathbb{R}^3$, i.e. $\mathbb{R}^3 - \{O\}$. That set is simply connected since any closed curve in it can be continuously contracted to a point while staying in the set. Since $\boldsymbol{F}$ has zero curl on a simply connected set, it is conservative. Indeed, $\varphi(x, y, z) = \frac{1}{2}\ln(x^2 + y^2 + z^2)$ is a potential.

14.8 Divergence Theorem

1. D
2. The divergence of $\boldsymbol{F}$ is 3. Therefore, by the divergence theorem, the flux of $\boldsymbol{F}$ through the indicated closed surface is 3 times the enclosed volume. The volume is a pyramid with base area 9 and height 2. Therefore, the flux is

$$3 \cdot \frac{1}{3} 9 \cdot 2 = 18$$

3.

   a. The divergence of $\boldsymbol{F}$ is 2.

   b. Let D be the closed unit disk in the xy plane. Then $D \cup S$ is a closed surface. It is a consequence of the divergence theorem that outflow of

F through S equals inflow of ***F*** through D + integral of div ***F*** over the enclosed volume.

The influx of ***F*** through D is easy to compute because $z = 0$ and $\boldsymbol{n} =< 0,0,1 >$ on D. Therefore, $\boldsymbol{F} \cdot \boldsymbol{n} = x^2 + y^2$ on D, which means that the influx of ***F*** through D is just the area integral of r^2 over the unit disk $r \leq 1$ in polar coordinates, which is $\int_0^{2\pi} \int_0^1 r^3 dr d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$.

The integral of div ***F*** over the enclosed volume is the integral of 2 over the upper half unit sphere, which is 2 times its volume, or $2 \cdot \frac{2}{3}\pi = \frac{4}{3}\pi$.

It follows that the upwards flux through S is $\frac{\pi}{2} + \frac{4\pi}{3} = \frac{11\pi}{6}$.

