Infinite Series – Some Tests for Convergence

A person with an ear infection is told to take antibiotic tablets regularly for several days. Since the drug is being excreted by the body between doses, how can we calculate the quantity of the drug remaining in the body at any particular time?

To be specific, let's suppose the drug is ampicillin (a common antibiotic) taken in 250 mg doses four times a day (that is, every six hours). It is known that at the end of six hours, about 4% of the drug is still in the body. What quantity of the drug is in the body right after the tenth tablet? The fortieth? The n-th?

Note:

The **sum of a finite geometric series** is given by

$$S_n = a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}, \quad \text{provided } x \neq 1.$$

For $|x| < 1$, the **sum of the infinite geometric series** is given by

$$S = a + ax + ax^2 + \cdots + ax^{n-1} + ax^n + \cdots = \frac{a}{1-x}.$$

1. For each of the following infinite geometric series, find several partial sums and the sum (if it exists).

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

(b) $1 + 2 + 4 + 8 + \cdots$

(c) $6 - 2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \cdots$

2. People who save money often do so by putting some fixed amount aside regularly. To be specific, suppose $1000 is deposited every year in a savings account earning 5% a year, compounded annually. What is the balance, B_n, in dollars, in the savings account right after the n^{th} deposit?

3. Find the sum of the series in the exercises below.

a. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^{10}}$

b. $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

c. $\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n$

d. $\sum_{n=4}^{20} \left(\frac{1}{3}\right)^n$

e. $1 + 3x + 9x^2 + 27x^3 + \cdots$

n-th partial sums test for divergence/convergence

If the sequence S_n of partial sums converges to S, so $\lim_{n\to\infty} S_n = S$, then we say the series $\sum_{n=1}^{\infty} a_n$ **converges** and that its sum is S. We write $\sum_{n=1}^{\infty} a_n = S$. If $\lim_{n\to\infty} S_n$ does not exist, we say that the series **diverges**.

Exercise: Test the convergence of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Convergence Properties of Series

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and if k is a constant, then
   - $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
   - $\sum_{n=1}^{\infty} ka_n$ converges to $k \sum_{n=1}^{\infty} a_n$.
2. Changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.
3. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
4. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ka_n$ diverges if $k \neq 0$.

Exercise: Does the series $\sum (1 - e^{-n})$ converge?

The Integral Test

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.

- If $\int_1^{\infty} f(x)dx$ converges, then $\sum a_n$ converges.
- If $\int_1^{\infty} f(x)dx$ diverges, then $\sum a_n$ diverges.

Exercise: For what values of p does the series $\sum_{n=1}^{\infty} 1/n^p$ converge?

Use the integral test to decide whether the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$
3. $\sum_{n=1}^{\infty} \frac{1}{e^n}$
4. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
5. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
6. $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$
7. $\sum_{n=1}^{\infty} \frac{n+2^n}{n2^n}$
8. $\sum_{n=1}^{\infty} \left(\left(\frac{3}{4}\right)^n + \frac{1}{n}\right)$
9. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n$
10. $\sum_{n=2}^{\infty} \frac{n \ln n + 4}{n^2}$

Comparison Test

Suppose $0 \le a_n \le b_n$ for all n beyond a certain value.

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

1. Use the comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges.
2. Decide whether the following series converge:

   a. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$

   b. $\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$

3. $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$

The Ratio Test

For a series $\sum a_n$, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

- If $L < 1$, then $\sum a_n$ converges.
- If $L > 1$, or if L is infinite,[5] then $\sum a_n$ diverges.
- If $L = 1$, the test does not tell us anything about the convergence of $\sum a_n$.

1. Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

2. Determine if the series converges or diverges: $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$

3. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$

4. $\sum_{n=1}^{\infty} \frac{n2^n}{3^n}$

The *root test* for convergence.

Given a series $\sum a_n$ of positive terms (that is, $a_n > 0$) such that the root $\sqrt[n]{a_n}$ has a limit r as $n \to \infty$,

- if $r < 1$, then $\sum a_n$ converges
- if $r > 1$, then $\sum a_n$ diverges
- if $r = 1$, then $\sum a_n$ could converge or diverge.

(This test works since $\lim_{n\to\infty} \sqrt[n]{a_n} = r$ tells us that the series is comparable to a geometric series with ratio r.) Use this test to determine the behavior of the series.

Test the following for convergence/divergence:

1. $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$

2. $\sum_{n=1}^{\infty} \left(\frac{5n+1}{3n^2}\right)^n$

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } c > 0,$$

then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

The limit $\lim_{n\to\infty} a_n/b_n = c$ captures the idea that a_n "behaves like" cb_n as $n \to \infty$.

Use the limit comparison test to determine if the following series converge or diverge.

(a) $\sum \frac{n^2+6}{n^4-2n+3}$

(b) $\sum \sin\left(\frac{1}{n}\right)$

Solution

(a)

We take $a_n = \frac{n^2+6}{n^4-2n+3}$. Because a_n behaves like $\frac{n^2}{n^4} = \frac{1}{n^2}$ as $n \to \infty$ we take $b_n = 1/n^2$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4+6n^2}{n^4-2n+3} = 1.$$

The limit comparison test applies with $c = 1$. Since $\sum 1/n^2$ converges, the limit comparison test shows that $\sum \frac{n^2+6}{n^4-2n+3}$ also converges.

(b)

Since $\sin(x) \approx x$ for x near 0, we know that $\sin\left(\frac{1}{n}\right)$ behaves like $1/n$ as $n \to \infty$. We apply the limit comparison test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

Thus $c = 1$ and since $\sum 1/n$ diverges, the series $\sum \sin(1/n)$ also diverges.