MAT 272

SPRING 2015

FINAL EXAM – Form A

SoMSS, ASU

Directions:

1. **There are 10 questions worth a total of 105 points.**
2. **Read all the questions carefully.**
3. **You must show all work in order to receive credit for the free response questions!!**
4. **When possible, box your answer, which must be complete, organized, and exact unless otherwise directed.**
5. **Always indicate how a calculator was used (i.e. sketch graph, etc. …).**
6. **No calculators with QWERTY keyboards or ones like TI-89 or TI-92 that do symbolic algebra may be used.**

Honor Statement:

By signing below you confirm that you have neither given nor received any unauthorized assistance on this exam. This includes any use of a graphing calculator beyond those uses specifically authorized by the Mathematics Department and your instructor. Furthermore, you agree not to discuss this exam with anyone until the exam testing period is over. In addition, your calculator's program memory and menus may be checked at any time and cleared by any testing center proctor or Mathematics Department instructor.

________________________________ Signature   Date ______________

PRINT NAME: ________________________________

PRINT TAs NAME: ________________________________

1. **[10 pts]** A vector field is given by $\mathbf{F}(x,y) = \langle 5x^4 y,\ x^5 - 8 \rangle$.
   a. Find a potential function for this vector field.

$$\phi(x,y) = x^5 y + g(y)$$

$$\frac{\partial \phi}{\partial y} = x^5 + g'(y) = x^5 - 8 \Rightarrow g'(y) = -8$$

$$\Rightarrow g(y) = -8y + k$$

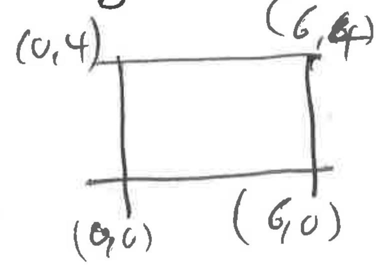
So, $\phi(x,y) = x^5 y - 8y + k$

   b. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the top half-circle $x^2 + y^2 = 4$ from $(-2,0)$ to $(2,0)$. Since $\bar{F}$ is conservative,

$$\int_C \bar{F} \cdot d\bar{r} = \phi(2,0) - \phi(-2,0) = 0$$

2. **[10 pts]** Use Green's Theorem to evaluate $\oint_C xe^{4x}\,dx - 6x^3 y\,dy$ where C is the rectangle from $(0,0)$ to $(6,0)$ to $(6,4)$ to $(0,4)$.

$M(x,y) = xe^{4x}$, $N(x,y) = -6x^3 y$



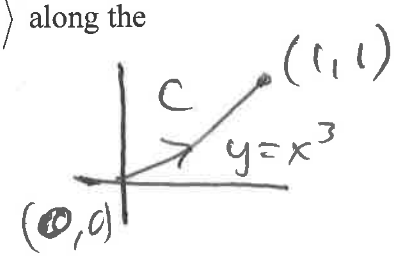
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -18x^2 y$$

$$\int_C xe^{4x}\,dx - 6x^3 y\,dy = \iint_R -18x^2 y\,dR$$

$$= \int_0^4 \int_0^6 -18x^2 y\,dx\,dy$$

$$= \int_0^6 \left(\int_0^4 y\,dy\right) \cdot \left(-18x^2\right) dx = \frac{-18(6^3)(16)}{6}$$

3. **[10 pts]** Compute the work done by the force field $\mathbf{F}(x,y)=\langle 0,x^3y^3\rangle$ along the curve C, where C is the portion of $y=x^3$ from $(0,0)$ to $(1,1)$



$$W=\int_C \bar{F}\cdot d\bar{r} = \int_{C:\,y=x^3} x^3y^3\,dy$$

$$= \int_0^1 y^4\,dy = \left.\frac{y^5}{5}\right|_0^1 = \frac{1}{5}$$

4. **[10 pts]**
   a. Parametrize the surface of the paraboloid $z=4-x^2-y^2$. Choose the natural parametrization.

$$\bar{r}(x,y)=\langle x, y, 4-x^2-y^2\rangle$$

   b. Use the parametrization obtained in part a. to set up the double integral, over the parametric domain, that represents the surface area of the paraboloid $z=4-x^2-y^2$, $z\geq 0$

$$\bar{r}_x=\langle 1,0,-2x\rangle, \quad \bar{r}_y=\langle 0,1,-2y\rangle$$

$$\bar{r}_x\times\bar{r}_y=\langle 2x,-2y,1\rangle$$

$$|\bar{r}_x\times\bar{r}_y|=\sqrt{1+4(x^2+y^2)}$$

$$S=\iint_R |\bar{r}_x\times\bar{r}_y|\,dR = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4(x^2+y^2)}\,dy\,dx.$$

5. **[10 pts]** Use the Divergence Theorem to compute $\iint_S \mathbf{F} \bullet \mathbf{n} dS$ for

$\mathbf{F} = \langle 5y^8, 2z - 9\cos x, 4z^3 + 7x \rangle$ and S is the boundary of the cube $-2 \le x \le 2$, $-2 \le y \le 2$ and $-2 \le z \le 2$.

$$\nabla \cdot \bar{F} = 0 + 0 + 12z^2$$

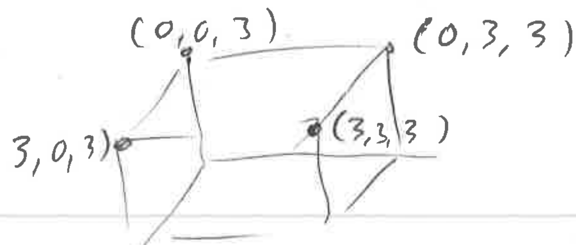
$$\iint_S \bar{F} \cdot \bar{n}\, dS = \iiint_V 12z^2\, dV$$

$$= \int_{-2}^{2}\int_{-2}^{2}\int_{-2}^{2} 12z^2\, dz\, dy\, dx = ?$$

do this

6. **[10 pts]** Use Stokes' Theorem to compute $\oint_C \mathbf{F} \bullet d\mathbf{r}$ for $\mathbf{F} = \langle x^5, y^3 + x, \cos z \rangle$ and C is the square $(0,0,3)$ to $(3,0,3)$ to $(3,3,3)$ to $(0,3,3)$ to $(0, 0, 3)$.

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, dS$$



$$\nabla \times \bar{F} = \langle 0, 0, 1 \rangle = \hat{k}$$

$$\bar{r}(x,y) = \langle x, y, 3 \rangle$$

$$\bar{r}_x = \langle 1, 0, 0 \rangle$$

$$\bar{r}_y = \langle 0, 1, 0 \rangle$$

$$\bar{r}_x \times \bar{r}_y = \langle 0, 0, 1 \rangle = \hat{k}$$

$$\iint_S \nabla \times \bar{F} \cdot \bar{n}\, dS$$

$$= \iint_R \hat{k} \cdot \hat{k}\, dx\, dy = \int_0^3\int_0^3 dx\, dy$$

$$= 9$$

$f(x,y,z) = z(x^2+y^2)$

7. **[10 pts]**
   a. At the point $P(-1,1,1)$, find the directional derivative of f in the direction $\mathbf{v} = <1,\ 2,\ -2>$.

$\nabla f = \langle 2xz,\ 2yz,\ x^2+y^2 \rangle$

$\nabla f(-1,1,1) = \langle -2, 2, 2 \rangle$

$\hat{v} = \langle \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \rangle$

$D_{\hat{v}} f(-1,1,1) = \langle -2, 2, 2 \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \rangle = \frac{-2}{3}$

   b. At the same point $P(-1,1,1)$, determine the value of the maximum rate of change of f, and the direction in which it occurs.

Max. directional derivative of $f = \sqrt{12}$.
in the direction $\langle -2, 2, 2 \rangle$.

8. **[10 pts]** What is the maximum of $f(x,y) = 4x + 6y$ where x and y are real numbers and $x^2 + y^2 = 10$?

$4 = 2x\lambda$, $6 = 2y\lambda$, $x^2+y^2 = 10$

$x = \frac{2}{\lambda}$, $y = \frac{3}{\lambda}$, $\frac{4}{\lambda^2} + \frac{9}{\lambda^2} = 10$

$\lambda = \pm\sqrt{1.3}$

Find these.

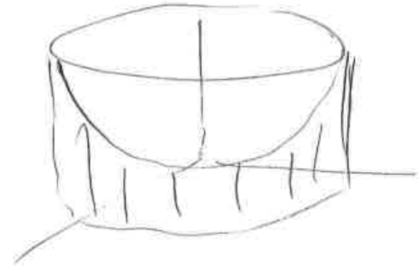
$x = \frac{2}{\sqrt{1.3}}$, $y = \frac{3}{\sqrt{1.3}}$ $f\left(\frac{2}{\sqrt{1.3}}, \frac{3}{\sqrt{1.3}}\right) = ?$

&

$x = \frac{-2}{\sqrt{1.3}}$, $y = \frac{-3}{\sqrt{1.3}}$ $f\left(\frac{-2}{\sqrt{1.3}}, \frac{-3}{\sqrt{1.3}}\right) = ?$

9. **[10 pts]** Use an appropriate coordinate system to compute the volume of the solid below $z = 5x^2 + 5y^2$, above $z = 0$ and inside $x^2 + y^2 = 9$.

$$V = \int_0^{2\pi} \left(\int_0^3 \left(\int_0^{5(x^2+y^2)} dz \right) r\,dr \right) d\theta$$



$$= \int_0^{2\pi} \int_0^3 5(x^2+y^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^3 5r^3\,dr\,d\theta = \int_0^{2\pi} \frac{405}{4}\,d\theta$$

$$= \frac{405}{4}(2\pi)$$

10. **[15 pts]** State whether the following statements are TRUE or FALSE. If FALSE, either correct the statement or give a counterexample:

a. If $\boldsymbol{F}$ is a vector field defined on $\mathbb{R}^3$ and $\nabla \times \boldsymbol{F} = \mathbf{0}$ everywhere, then $\boldsymbol{F}$ may or may not be conservative.

False, F is conservative.

b. By reversing the order of integration $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dxdy = \int_{\sqrt{y}}^1 \int_0^1 \sqrt{x^3+1}\, dydx$.

False, $\int_0^1 \left(\int_0^{x^2} \sqrt{x^3+1}\, dy \right) dx$

c. If the directional derivative $D_{\boldsymbol{u}} f$ of a function of two variables f is zero at $(\boldsymbol{a}, \boldsymbol{b})$ for all unit vectors $\boldsymbol{u}$, then, f must have a local minimum or maximum at (a,b).

False, Could be a saddle point.

d. The circulation of the vector field $\boldsymbol{F}(x,y) = \langle y^2, 2xy \rangle$ over the counter-clockwise unit circle is 1.

$$\int_{C: x^2+y^2=1} y^2 dx + 2xy\, dy = \int_0^{2\pi} [\sin^2 t(-\sin t) + 2\cos t \sin t \cos t] dt = 0$$

e. A parametric representation of the surface $y^2 + z^2 = 16$ from $x = -8$ to $x = 8$ is $-8 \le x \le 8$, $y = 4\cos\theta$, $z = 4\sin\theta$, $0 \le \theta \le 2\pi$

True,