MAT 242 Test 3 SOLUTIONS, FORM A

1. Let $\vec{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 2 \\ -2 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} -2 & 4 & 1 \\ 2 & 1 & 4 \\ 1 & -2 & 2 \\ 4 & 2 & -2 \end{bmatrix}$$

a. [15 points] Find the orthogonal projection of $\begin{bmatrix} -13 \\ 18 \\ 9 \\ 1 \end{bmatrix}$ into W, without inverting any matrices or solving any systems of linear equations.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{75}{25} = 3,$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{-50}{25} = -2,$$

$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{75}{25} = 3,$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} -11 \\ 16 \\ 13 \\ 2 \end{bmatrix}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1}A^\top\vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\vec{v}_1 \cdot \vec{v}_1 = 25$$

$$\vec{v}_2 \cdot \vec{v}_2 = 25$$

$$\vec{v}_3 \cdot \vec{v}_3 = 25$$

$$\left\{ \begin{bmatrix} 4/5 \\ 1/5 \\ -2/5 \\ 2/5 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 4/5 \\ 2/5 \\ -2/5 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 2/5 \\ 1/5 \\ 4/5 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM A

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} 0 \\ -2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \\ 7 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

   a. [15 points] Find the vector in W closest to $\begin{bmatrix} -1 \\ 7 \\ -5 \\ -4 \end{bmatrix}$.

$$\vec{c} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 0 \\ 7 \\ -5 \\ -4 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: −5 points for only finding the coordinates.

   b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} 0 \\ -2 \\ -2 \\ -1 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ -3 \end{bmatrix} - \frac{9}{9}\begin{bmatrix} 0 \\ -2 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \\ 7 \end{bmatrix} - \frac{-27}{9}\begin{bmatrix} 0 \\ -2 \\ -2 \\ -1 \end{bmatrix} - \frac{-9}{9}\begin{bmatrix} 0 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 2 \end{bmatrix} \blacksquare$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM A

3. Do the following, for the following set of data points: $(-4, -93)$, $(-1, -3)$, $(0, -1)$, $(4, 27)$.

   a. [10 points] Find the parabola $y = ax^2 + bx + c$ which best fits these points.

$$A = \begin{bmatrix} 16 & -4 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 16 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -93 \\ -3 \\ -1 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -2393/979 \\ 14445/979 \\ 6221/979 \end{bmatrix}$$

$$y = \frac{-2393}{979}x^2 + \frac{14445}{979}x + \frac{6221}{979} = (-2.444331)\,x^2 + (14.754852)\,x + 6.354443$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

   b. [10 points] Find the parabola $y = ax^2 + c$ with no linear term which best fits these points.

$$A = \begin{bmatrix} 16 & 1 \\ 1 & 1 \\ 0 & 1 \\ 16 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -93 \\ -3 \\ -1 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$y = -2x^2 - 1 = (-2.000000)\,x^2 - 1.000000$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

MAT 242 Test 3 SOLUTIONS, FORM A

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} -x_1 & + & 5x_2 & - & 3x_3 & = & -2 \\ 2x_1 & + & 3x_2 & - & 4x_3 & = & 0 \\ 3x_1 & + & x_2 & - & 3x_3 & = & 2 \\ 3x_1 & - & 4x_2 & + & 5x_3 & = & -4 \end{array}$$

$$\hat{x} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} -141/217 \\ -342/217 \\ -361/217 \end{bmatrix} = \begin{bmatrix} -0.649770 \\ -1.576037 \\ -1.663594 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} -4 \\ 2 \\ -4 \\ -4 \end{bmatrix} \right\}$$

$$A^\top = [\,-4 \quad 2 \ -4 \ -4\,] \ \overrightarrow{\text{RREF}} \ [\,1 \ -1/2 \quad 1 \quad 1\,]$$

$$-4x_1 + 2x_2 - 4x_3 - 4x_4 = 0$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM B

1. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & 0 \end{bmatrix}$$

a. [15 points] Find the vector in W closest to $\begin{bmatrix} -5 \\ 4 \\ -1 \\ 7 \end{bmatrix}$, without inverting any matrices or solving any systems of linear equations.

$$\begin{aligned} c_1 &= \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{-27}{9} = -3, \\ c_2 &= \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{0}{9} = 0, \\ c_3 &= \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{-1}{1} = -1, \\ \vec{p} = A\vec{c} &= \begin{bmatrix} -3 \\ 6 \\ -1 \\ 6 \end{bmatrix} \end{aligned}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1} A^\top \vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_1 &= 9 \\ \vec{v}_2 \cdot \vec{v}_2 &= 9 \\ \vec{v}_3 \cdot \vec{v}_3 &= 1 \end{aligned}$$

$$\left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ -2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 0 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM B

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} -2 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -15 \\ 0 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

a. [15 points] Find the vector in W closest to $\begin{bmatrix} 6 \\ -12 \\ -8 \\ 9 \end{bmatrix}$.

$$\vec{c} = \left(A^\top A\right)^{-1} \left(A^\top B\right) = \begin{bmatrix} 4 \\ -8 \\ -2 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 2 \\ -14 \\ -6 \\ 8 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: −5 points for only finding the coordinates.

b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} -2 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} 0 \\ 5 \\ 5 \\ 0 \end{bmatrix} - \frac{25}{25} \begin{bmatrix} -2 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -2 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} -5 \\ -5 \\ -15 \\ 0 \end{bmatrix} - \frac{-25}{25} \begin{bmatrix} -2 \\ 4 \\ 1 \\ 2 \end{bmatrix} - \frac{-75}{25} \begin{bmatrix} 2 \\ 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ -4 \end{bmatrix}$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM B

3. Do the following, for the following set of data points: $(-5,-133)$, $(-4,-71)$, $(0,-3)$, $(3,27)$.

   a. [10 points] Find the parabola $y = ax^2 + bx + c$ which best fits these points.

$$A = \begin{bmatrix} 25 & -5 & 1 \\ 16 & -4 & 1 \\ 0 & 0 & 1 \\ 9 & 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -133 \\ -71 \\ -3 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -1821/781 \\ 10613/781 \\ 3537/781 \end{bmatrix}$$

$$y = \frac{-1821}{781}x^2 + \frac{10613}{781}x + \frac{3537}{781} = (-2.331626)\,x^2 + (13.588988)\,x + 4.528809$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

   b. [10 points] Find the parabola $y = ax^2 + c$ with no linear term which best fits these points.

$$A = \begin{bmatrix} 25 & 1 \\ 16 & 1 \\ 0 & 1 \\ 9 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -133 \\ -71 \\ -3 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -1968/337 \\ 9435/337 \end{bmatrix}$$

$$y = \frac{-1968}{337}x^2 + \frac{9435}{337} = (-5.839763)\,x^2 + 27.997033$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

MAT 242 Test 3 SOLUTIONS, FORM B

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} 4x_1 & + & 5x_2 & & & = & -3 \\ x_1 & + & x_2 & + & x_3 & = & 0 \\ & & & - & 4x_3 & = & 2 \\ -3x_1 & + & 3x_2 & + & 2x_3 & = & -7 \end{array}$$

$$\hat{x} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 1544/1871 \\ -16015/13097 \\ -5809/13097 \end{bmatrix} = \begin{bmatrix} 0.825227 \\ -1.222799 \\ -0.443537 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -4 \\ 4 \end{bmatrix} \right\}$$

$$A^\top = \begin{bmatrix} 0 & 1 & -4 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 0 & 1 & -4 & 4 \end{bmatrix}$$

$$x_2 - 4x_3 + 4x_4 = 0$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM C

1. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & -2 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

a. [15 points] Find the vector in W closest to $\begin{bmatrix} 2 \\ 1 \\ -2 \\ -11 \end{bmatrix}$, without inverting any matrices or solving any systems of linear equations.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{18}{9} = 2,$$
$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{27}{9} = 3,$$
$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{2}{1} = 2,$$
$$\vec{p} = A\vec{c} = \begin{bmatrix} 2 \\ -1 \\ -4 \\ -10 \end{bmatrix}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1} A^\top \vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\vec{v}_1 \cdot \vec{v}_1 = 9$$
$$\vec{v}_2 \cdot \vec{v}_2 = 9$$
$$\vec{v}_3 \cdot \vec{v}_3 = 1$$
$$\left\{ \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM C

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ -1 \\ -1 \\ -4 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

   a. [15 points] Find the orthogonal projection of $\begin{bmatrix} 1 \\ -4 \\ 1 \\ 3 \end{bmatrix}$ into W.

$$\vec{c} = \left(A^\top A\right)^{-1} \left(A^\top B\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: -5 points for only finding the coordinates.

   b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} -3 \\ -1 \\ 2 \\ 2 \end{bmatrix} - \frac{-9}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} 9 \\ -1 \\ -1 \\ -4 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix} - \frac{-27}{9} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \blacksquare$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM C

3. Do the following, for the following set of data points: $(-5,-10)$, $(-4,12)$, $(0,0)$, $(2,18)$.

a. [10 points] Find the line $y = ax + b$ which best fits these points.

$$A = \begin{bmatrix} -5 & 1 \\ -4 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -10 \\ 12 \\ 0 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 292/131 \\ 1166/131 \end{bmatrix}$$

$$y = \frac{292}{131}x + \frac{1166}{131} = (2.229008)\,x + 8.900763$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

b. [10 points] Find the parabola $y = ax^2 + c$ with no linear term which best fits these points.

$$A = \begin{bmatrix} 25 & 1 \\ 16 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -10 \\ 12 \\ 0 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -844/1563 \\ 5770/521 \end{bmatrix}$$

$$y = \frac{-844}{1563}x^2 + \frac{5770}{521} = (-0.539987)\,x^2 + 11.074856$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

MAT 242 Test 3 SOLUTIONS, FORM C

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} -2x_1 & + & 2x_2 & + & 4x_3 & = & 5 \\ -x_1 & + & 4x_2 & + & 5x_3 & = & 0 \\ x_1 & + & 5x_2 & + & 3x_3 & = & 0 \\ -x_1 & + & x_2 & & & = & -5 \end{array}$$

$$\hat{x} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 785/399 \\ -940/399 \\ 375/133 \end{bmatrix} = \begin{bmatrix} 1.967419 \\ -2.355890 \\ 2.819549 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \\ 3 \end{bmatrix} \right\}$$

$$A^\top = \begin{bmatrix} 2 & 2 & -2 & 1 \\ -4 & 0 & 2 & -4 \\ -2 & 2 & -2 & 3 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = & 0 \\ -4x_1 & & & + & 2x_3 & - & 4x_4 & = & 0 \\ -2x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & = & 0 \end{array}$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1/2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points for the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM D

1. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -2 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -2 \\ -1 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}$$

a. [15 points] Find the orthogonal projection of $\begin{bmatrix} 4 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ into W, without inverting any matrices or solving any systems of linear equations.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{2}{1} = 2,$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{9}{9} = 1,$$

$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{0}{9} = 0,$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 2 \end{bmatrix}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1}A^\top\vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\vec{v}_1 \cdot \vec{v}_1 = 1$$

$$\vec{v}_2 \cdot \vec{v}_2 = 9$$

$$\vec{v}_3 \cdot \vec{v}_3 = 9$$

$$\left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ -2/3 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM D

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -2 \\ -14 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -11 \\ -2 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

   a. [15 points] Find the vector in W closest to $\begin{bmatrix} 3 \\ -6 \\ -7 \\ -9 \end{bmatrix}$.

$$\vec{c} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 1 \\ -2 \\ -9 \\ -8 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: −5 points for only finding the coordinates.

   b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} -7 \\ -1 \\ -2 \\ -14 \end{bmatrix} - \frac{-75}{25}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -2 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} 4 \\ -3 \\ -11 \\ -2 \end{bmatrix} - \frac{-25}{25}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \frac{-50}{25}\begin{bmatrix} -1 \\ 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM D

3. Do the following, for the following set of data points: $(-3, 51)$, $(-2, 20)$, $(0, 0)$, $(4, -40)$.

   a. [10 points] Find the line $y = ax + b$ which best fits these points.

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 51 \\ 20 \\ 0 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -1381/115 \\ 546/115 \end{bmatrix}$$

$$y = \frac{-1381}{115}x + \frac{546}{115} = (-12.008696)\,x + 4.747826$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

   b. [10 points] Find the parabola $y = ax^2 + c$ with no linear term which best fits these points.

$$A = \begin{bmatrix} 9 & 1 \\ 4 & 1 \\ 0 & 1 \\ 16 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 51 \\ 20 \\ 0 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -1303/571 \\ 13872/571 \end{bmatrix}$$

$$y = \frac{-1303}{571}x^2 + \frac{13872}{571} = (-2.281961)\,x^2 + 24.294221$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

MAT 242 Test 3 SOLUTIONS, FORM D

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & 5x_3 & = & 7 \\ x_1 & + & 3x_2 & - & x_3 & = & 5 \\ -x_1 & + & x_2 & - & 4x_3 & = & 2 \\ & - & 2x_2 & - & 5x_3 & = & -2 \end{array}$$

$$\hat{x} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 647/303 \\ 3962/3333 \\ -733/3333 \end{bmatrix} = \begin{bmatrix} 2.135314 \\ 1.188719 \\ -0.219922 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$A^\top = \begin{bmatrix} -3 & 0 & 0 & -1 \\ 4 & 1 & 3 & -2 \end{bmatrix} \xrightarrow{\hspace{2em}} \text{RREF} \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 3 & -10/3 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} -3x_1 & & & & & - & x_4 & = & 0 \\ 4x_1 & + & x_2 & + & 3x_3 & - & 2x_4 & = & 0 \end{array}$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -1/3 \\ 10/3 \\ 0 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM E

1. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. [15 points] Find the vector in W closest to $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$, without inverting any matrices or solving any systems of linear equations.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{2}{1} = 2,$$
$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{3}{1} = 3,$$
$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{0}{1} = 0,$$
$$\vec{p} = A\vec{c} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1} A^\top \vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\vec{v}_1 \cdot \vec{v}_1 = 1$$
$$\vec{v}_2 \cdot \vec{v}_2 = 1$$
$$\vec{v}_3 \cdot \vec{v}_3 = 1$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM E

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

a. [15 points] Find the orthogonal projection of $\begin{bmatrix} -1 \\ -1 \\ -10 \\ -5 \end{bmatrix}$ into W.

$$\vec{c} = \left(A^\top A\right)^{-1} \left(A^\top B\right) = \begin{bmatrix} -6 \\ -3 \\ 2 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 0 \\ -1 \\ -10 \\ -5 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: −5 points for only finding the coordinates.

b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} 0 \\ 10 \\ 1 \\ 5 \end{bmatrix} - \frac{27}{9} \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \frac{18}{9} \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix} \blacksquare$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM E

3. Do the following, for the following set of data points: $(-1, 11)$, $(1, 3)$, $(2, 20)$, $(4, 126)$.

   a. [10 points] Find the parabola $y = ax^2 + bx + c$ which best fits these points.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 11 \\ 3 \\ 20 \\ 126 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 19/2 \\ -149/26 \\ -95/26 \end{bmatrix}$$

$$y = \frac{19}{2}x^2 + \frac{-149}{26}x - \frac{95}{26} = (9.500000)\,x^2 + (-5.730769)\,x - 3.653846$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

   b. [10 points] Find the parabola $y = ax^2 + bx$ passing through the origin which best fits these points.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 4 & 2 \\ 16 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 11 \\ 3 \\ 20 \\ 126 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 1957/211 \\ -1264/211 \end{bmatrix}$$

$$y = \frac{1957}{211}x^2 + \frac{-1264}{211}x = (9.274882)\,x^2 + (-5.990521)\,x$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

MAT 242 Test 3 SOLUTIONS, FORM E

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} 4x_1 & & & + & 2x_3 & = & 5 \\ -x_1 & + & 5x_2 & & & = & -7 \\ -5x_1 & + & 4x_2 & - & 2x_3 & = & 3 \\ -3x_1 & - & x_2 & & & = & -2 \end{array}$$

$$\hat{x} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 238/341 \\ -285/682 \\ -1015/682 \end{bmatrix} = \begin{bmatrix} 0.697947 \\ -0.417889 \\ -1.488270 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix} \right\}$$

$$A^\top = \begin{bmatrix} 0 & -1 & -2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 0 & 1 & 2 & -2 \end{bmatrix}$$

$$-\ x_2 - 2x_3 + 2x_4 = 0$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM F

1. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

a. [15 points] Find the orthogonal projection of $\begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}$ into W, without inverting any matrices or solving any systems of linear equations.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{0}{1} = 0,$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{-3}{1} = -3,$$

$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{-2}{1} = -2,$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1} A^\top \vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\vec{v}_1 \cdot \vec{v}_1 = 1$$

$$\vec{v}_2 \cdot \vec{v}_2 = 1$$

$$\vec{v}_3 \cdot \vec{v}_3 = 1$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM F

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -10 \\ 7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ -1 \\ -2 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

a. [15 points] Find the vector in W closest to $\begin{bmatrix} 3 \\ 1 \\ -9 \\ -6 \end{bmatrix}$.

$$\vec{c} = \left(A^\top A\right)^{-1} \left(A^\top B\right) = \begin{bmatrix} 34 \\ 8 \\ -3 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 3 \\ 0 \\ -9 \\ -6 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: −5 points for only finding the coordinates.

b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} 2 \\ 0 \\ -10 \\ 7 \end{bmatrix} - \frac{-36}{9} \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \\ -1 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} -7 \\ 0 \\ -1 \\ -2 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \end{bmatrix} - \frac{18}{9} \begin{bmatrix} -2 \\ 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \blacksquare$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM F

3. Do the following, for the following set of data points: $(-5, 110)$, $(-3, 26)$, $(0, 5)$, $(4, -79)$.

a. [10 points] Find the parabola $y = ax^2 + bx + c$ which best fits these points.

$$A = \begin{bmatrix} 25 & -5 & 1 \\ 9 & -3 & 1 \\ 0 & 0 & 1 \\ 16 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 110 \\ 26 \\ 5 \\ -79 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 553/781 \\ -14518/781 \\ -9325/781 \end{bmatrix}$$

$$y = \frac{553}{781}x^2 + \frac{-14518}{781}x - \frac{9325}{781} = (0.708067)\,x^2 + (-18.588988)\,x - 11.939821$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

b. [10 points] Find the parabola $y = ax^2 + bx$ passing through the origin which best fits these points.

$$A = \begin{bmatrix} 25 & -5 \\ 9 & -3 \\ 0 & 0 \\ 16 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 110 \\ 26 \\ 5 \\ -79 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 244/3363 \\ -63064/3363 \end{bmatrix}$$

$$y = \frac{244}{3363}x^2 + \frac{-63064}{3363}x = (0.072554)\,x^2 + (-18.752304)\,x$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

MAT 242 Test 3 SOLUTIONS, FORM F

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{aligned} -5x_1 - x_2 - 2x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= -7 \\ 5x_1 + 5x_2 - 2x_3 &= -6 \\ -3x_1 - x_2 + 3x_3 &= -6 \end{aligned}$$

$$\hat{x} = \left(A^\top A\right)^{-1} \left(A^\top B\right) = \begin{bmatrix} 2849/4377 \\ -26779/8754 \\ -7241/2918 \end{bmatrix} = \begin{bmatrix} 0.650902 \\ -3.059059 \\ -2.481494 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} -1 \\ 4 \\ 0 \\ 4 \end{bmatrix} \right\}$$

$$A^\top = \begin{bmatrix} -1 & 4 & 0 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -4 & 0 & -4 \end{bmatrix}$$

$$-x_1 + 4x_2 + 4x_4 = 0$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM MAKE-UP

1. Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ -1 \\ 2 \\ -2 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 4 \end{bmatrix}$. Note that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Also, let W be the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$A = \begin{bmatrix} 2 & -4 & -2 \\ -2 & -1 & 2 \\ 4 & 2 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

a. [15 points] Find the vector in W closest to $\begin{bmatrix} -7 \\ -13 \\ 6 \\ -11 \end{bmatrix}$, without inverting any matrices or solving any systems of linear equations.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{25}{25} = 1,$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{75}{25} = 3,$$

$$c_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{-50}{25} = -2,$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} -6 \\ -9 \\ 8 \\ -13 \end{bmatrix}$$

Grading: +10 points for finding the c_is, +5 points for finding $\vec{p}$. Grading for common mistakes: −5 points for using the $(A^\top A)^{-1} A^\top \vec{u}$ formula.

b. [10 points] Find an orthonormal basis for W.

$$\vec{v}_1 \cdot \vec{v}_1 = 25$$

$$\vec{v}_2 \cdot \vec{v}_2 = 25$$

$$\vec{v}_3 \cdot \vec{v}_3 = 25$$

$$\left\{ \begin{bmatrix} -2/5 \\ 2/5 \\ 1/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ -1/5 \\ 2/5 \\ -2/5 \end{bmatrix}, \begin{bmatrix} 2/5 \\ -2/5 \\ 4/5 \\ 1/5 \end{bmatrix} \right\}$$

Grading: +5 points for knowing to divide a vector by its length, +5 points for doing it with the orthogonal basis. Grading for common mistakes: +5 points (total) for doing Gram-Schmidt.

MAT 242 Test 3 SOLUTIONS, FORM MAKE-UP

2. Let W be the subspace spanned by $\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -3 \\ 0 \end{bmatrix} \right\}$. Note that this basis is **not** orthogonal.

a. [15 points] Find the orthogonal projection of $\begin{bmatrix} 6 \\ 0 \\ 1 \\ -3 \end{bmatrix}$ into W.

$$\vec{c} = \left(A^\top A\right)^{-1} \left(A^\top B\right) = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{p} = A\vec{c} = \begin{bmatrix} 4 \\ -2 \\ 1 \\ -4 \end{bmatrix}$$

Grading: +3 points for finding A and $\vec{u}$, +3 points for the formula, +4 points for the calculation. Grading for common mistakes: −5 points for only finding the coordinates.

b. [15 points] Find an orthogonal basis for W.

$$NEW_1 = OLD_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$NEW_2 = OLD_2 - \frac{OLD_2 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{-1}{1} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$NEW_3 = OLD_3 - \frac{OLD_3 \cdot NEW_1}{NEW_1 \cdot NEW_1} NEW_1 - \frac{OLD_3 \cdot NEW_2}{NEW_2 \cdot NEW_2} NEW_2 = \begin{bmatrix} -3 \\ 3 \\ -3 \\ 0 \end{bmatrix} - \frac{3}{1} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \end{bmatrix} \blacksquare$$

Grading: +5 points for each vector. Grading for common mistakes: +5 points (total) for dividing each vector by its length.

MAT 242 Test 3 SOLUTIONS, FORM MAKE-UP

3. Do the following, for the following set of data points: $(-5,5)$, $(-2,17)$, $(-1,9)$, $(4,149)$.

   a. [10 points] Find the parabola $y = ax^2 + bx + c$ which best fits these points.

$$A = \begin{bmatrix} 25 & -5 & 1 \\ 4 & -2 & 1 \\ 1 & -1 & 1 \\ 16 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ 17 \\ 9 \\ 149 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 2148/829 \\ 15202/829 \\ 27805/829 \end{bmatrix}$$

$$y = \frac{2148}{829}x^2 + \frac{15202}{829}x + \frac{27805}{829} = (2.591074)\,x^2 + (18.337756)\,x + 33.540410$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

   b. [10 points] Find the parabola $y = ax^2 + c$ with no linear term which best fits these points.

$$A = \begin{bmatrix} 25 & 1 \\ 4 & 1 \\ 1 & 1 \\ 16 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ 17 \\ 9 \\ 149 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 172/123 \\ 3557/123 \end{bmatrix}$$

$$y = \frac{172}{123}x^2 + \frac{3557}{123} = (1.398374)\,x^2 + 28.918699$$

Grading: +3 points for A, +2 points for B, +3 points for finding the coefficients, +2 points for the equation $y = \cdots$.

4. [10 points] Find the Least Squares Solution to the following system of linear equations:

$$\begin{array}{rcrcrcr} -x_1 & + & 2x_2 & + & 2x_3 & = & -1 \\ -4x_1 & - & 2x_2 & - & 4x_3 & = & -3 \\ -2x_1 & - & 4x_2 & - & 5x_3 & = & 2 \\ -5x_1 & - & 4x_2 & & & = & 3 \end{array}$$

$$\hat{x} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 1884/3215 \\ -4833/3215 \\ 2432/3215 \end{bmatrix} = \begin{bmatrix} 0.586003 \\ -1.503266 \\ 0.756454 \end{bmatrix}$$

Grading: +5 points for writing down A and B, +2 points for the normal equation, +3 points for the calculation. Grading for common mistakes: −3 points for finding $A\hat{x}$.

5. [15 points] Find a basis for $W^\perp$, the orthogonal complement of W, if W is the subspace spanned by

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -2 \\ 0 \end{bmatrix} \right\}$$

$$A^\top = \begin{bmatrix} -3 & -3 & -2 & 0 \\ -1 & -2 & 0 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 4/3 & 4 \\ 0 & 1 & -2/3 & -4 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} -3x_1 & - & 3x_2 & - & 2x_3 & & & = & 0 \\ -x_1 & - & 2x_2 & & & + & 4x_4 & = & 0 \end{array}$$

Parameterized by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \cdot \begin{bmatrix} -4/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -4 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Grading: +5 points for $A^\top$, +10 points the null space basis. Grading for common mistakes: +5 points (total) for Gram-Schmidt.