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Course: APM 504

Program: Mathematics MA

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APM 504 Project

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Section 8.3 Summary:

This section of the book focuses on calculating the escape probability ($p_{esc}(a, b)$) where a and b are vertices on an electrical networks. Significantly, a parallel is drawn between electric networks and random walks. There are two methods drawn from physics that can be used to calculate these escape probabilities. One method is based on simplifying the network and finding the current. Another method is based on finding the unique harmonic function that represents the voltage of the electric network.

Exercise 8.6.

Consider the electrical network with resistance shown in Figure 8.11, and put a battery that establishes a voltage one at a and zero at b.

1. Find the voltage at vertices c and d.

Proof: To solve for the voltage running through vertex c and d, we first need to simplify the circuit to obtain the equivalent resistance. With this, we will be able to find the current of the circuit and thus solve for the voltage at vertex c and d.

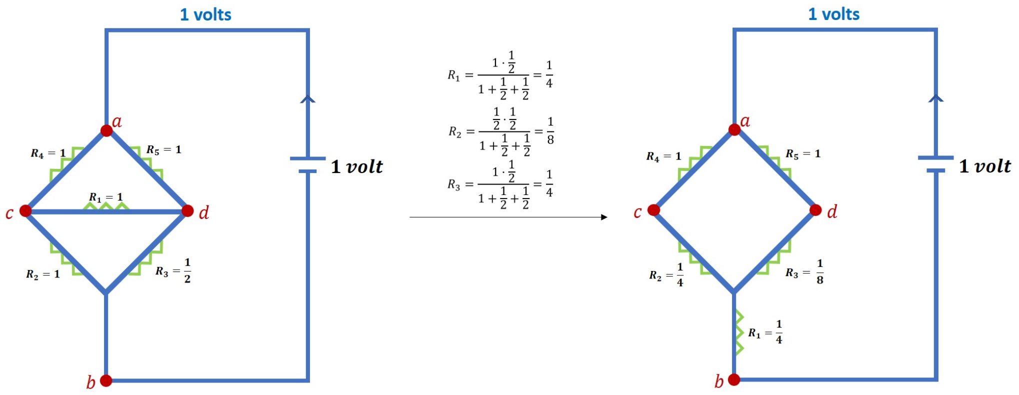


Figure 1: Initial and Intermediate circuit

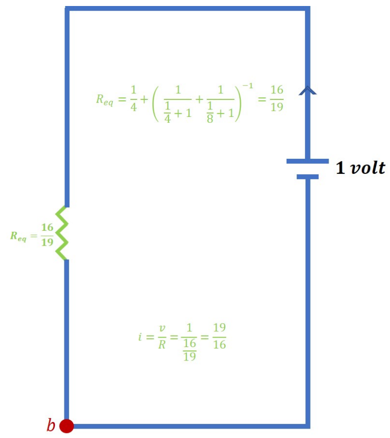


Figure 2: Simplified circuit

As seen in Figure 2, the effective resistance of the circuit is $R_{eff} = \frac{16}{19}$, hence we get $i(a) = \frac{19}{16}$. With this information and the intermediate circuit, we solve for the voltage and current at each vertex and resistor as seen in the table below.

	R	i	Volts	Volts#
R_1	$\frac{1}{4}$	$\frac{19}{16}$	$\frac{1}{4}\cdot\frac{19}{16}$	$\frac{4}{19}$
a	0	$\frac{19}{16}$	1	1
b	0	$\frac{19}{16}$	$1-\left(1-\frac{1}{4}\cdot\frac{19}{16}\right)-\frac{1}{4}\cdot\frac{19}{16}$	0
R_2	$\frac{1}{4}$	$\frac{4}{5}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{1}{5}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{9}{64}$
R_3	$\frac{1}{8}$	$\frac{8}{9}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{1}{9}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{5}{64}$
R_4	1	$\frac{4}{5}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{4}{5}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{9}{16}$
R_5	1	$\frac{8}{9}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{8}{9}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{5}{8}$
c	0	$\frac{4}{5}(1-\frac{1}{4}\cdot\frac{19}{16})$	$1-\frac{4}{5}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{7}{16}$
d	0	$\frac{8}{9}(1-\frac{1}{4}\cdot\frac{19}{16})$	$1-\frac{8}{9}(1-\frac{1}{4}\cdot\frac{19}{16})$	$\frac{3}{8}$

Figure 3: The table shows the resistance, voltage and, current of each vertex and resistor.

2. Deduce the probability that the random walk on this electrical network reaches vertex b before returning to a when starting from vertex a.

Proof: Recall from part a) we have $i(a) = \frac{19}{16}$, so by Lemma 8.5, we conclude that the probability that the symmetric random walk starting from a reaches b before returning a is

$$p_{esc}(a,b) = \frac{c_{eff}(a,b)}{c(a)} = \frac{i(a)}{c(a)} = \frac{19}{16}\cdot\frac{1}{2} = \frac{19}{32} \approx 0.5938$$

Exercise 8.7.

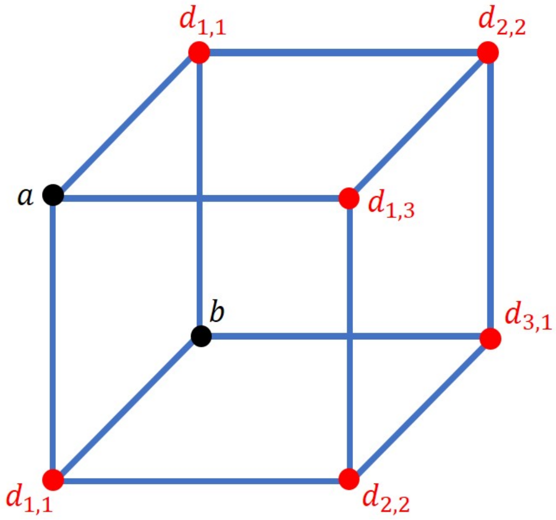
Consider the electrical network obtained from the cube by assigning resistance one to each of the twelve edges. Let a and b be two vertices at graph distance two from each other and let h be the unique harmonic function taking the value one at a and zero at b.

1. Use a probabilistic argument to find (without calculation) the value that the function h takes at each of the vertices x such that $d(a,x) = d(b,x)$.

Argument: When $d(a,x) = d(b,x)$, $h(x) = \frac{1}{2}$. Since our cube is symmetric, when $d(a,x) = d(b,x)$ we know that for each path from a to x there is a symmetric path from b to x. Then, since $h(a) = 1$ and $h(b) = 0$, and x is a symmetric center between a and b, $h(x) = \frac{1}{2}$.

2. Find more generally the of h at all vertices.

Proof: In the figure below, notice the naming scheme of the vertices $d_{i,j}$ for $i, j = 1, 2, 3$ where i is the distance from vertex a and, j the distance from vertex b.



Since the voltage function is the unique harmonic function on the network with $h(a) = 1$ and $h(b) = 0$ we can write the following system:

$$h(d_{1,1}) = \frac{h(a) + h(b) + h(d_{2,2})}{3}$$
$$h(d_{2,2}) = \frac{h(d_{1,1}) + h(d_{3,1}) + h(d_{1,3})}{3}$$
$$h(d_{1,3}) = \frac{h(a) + 2h(d_{2,2})}{3}$$
$$h(d_{3,1}) = \frac{h(b) + 2h(d_{2,2})}{3}$$

With some algebra we arrive with,

$$h(d_{2,2}) = \frac{1}{2}, \quad h(d_{3,1}) = \frac{1}{3} \quad h(d_{1,3}) = \frac{2}{3} \quad h(d_{1,1}) = \frac{1}{2}$$

3. Deduce that the probability that the symmetric random walk on the cube starting at vertex a reaches b before returning to a is equal to $4/9$.

Proof: The figures below outline how to solve the circuit for the effective resistance between vertices a and b. Note all edges of the networks below have a resistance of one unless otherwise specified.

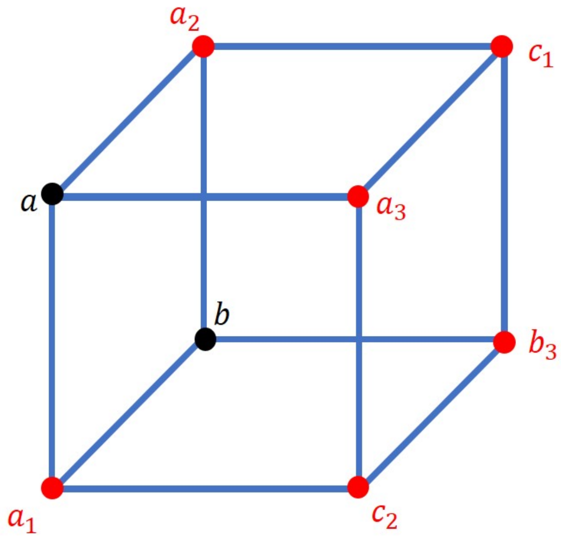


Figure 4: Cube.

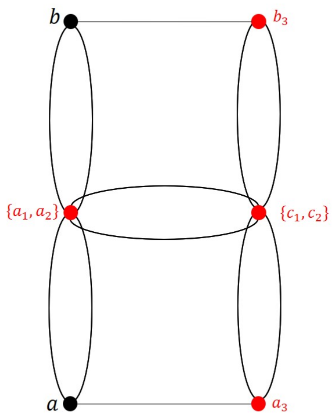


Figure 5: Initial circuit

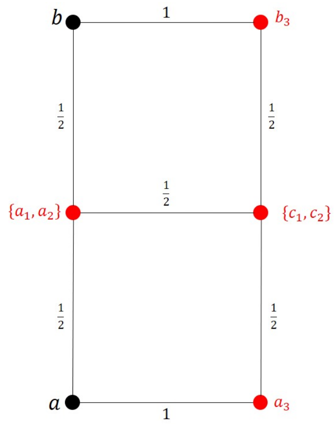


Figure 6: Intermediate circuit

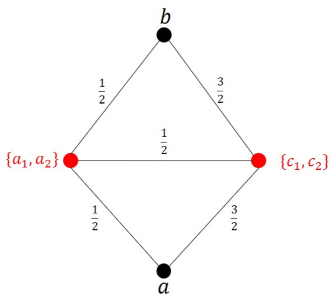
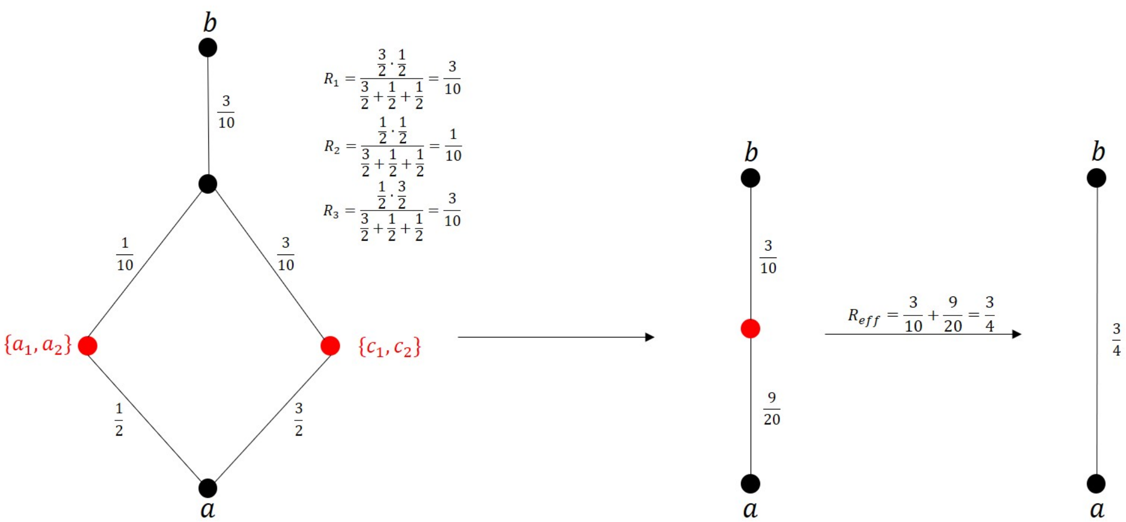


Figure 7: Simplified circuit



So from the simplified circuit above, we find $R_{eff} = \frac{3}{4}$, therefore we have $i(a) = \frac{4}{3}$, so by Lemma 8.5, we conclude that the probability that the symmetric random walk starting from a reaches b before returning a is

$$p_{esc}(a, b) = \frac{c_{eff}(a, b)}{c(a)} = \frac{i(a)}{c(a)} = \frac{4}{3} \cdot \frac{1}{3} = \frac{4}{9}$$

Exercise 8.8.

Let a and b be two adjacent vertices of the electrical network obtained from the cube by assigning resistance one to each of the twelve edges.

1. Use the two rules shown in Figure 8.4 as in Example 8.1 to find the effective resistance between a and b.

Proof: The figures below outline how to solve the circuit for the effective resistance between vertices a and b. Note all edges of the networks below have a resistance of one unless otherwise specified.

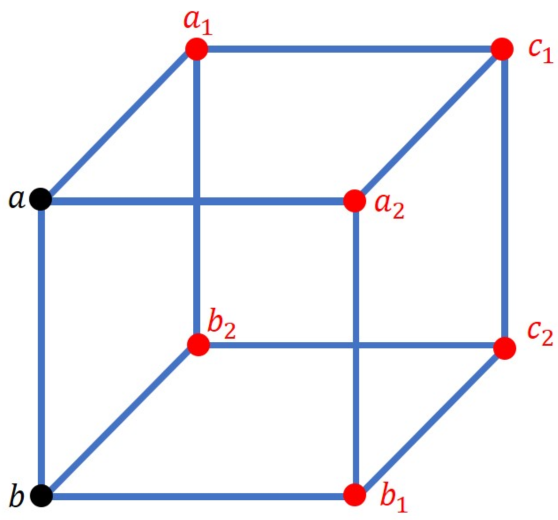


Figure 8: Cube.

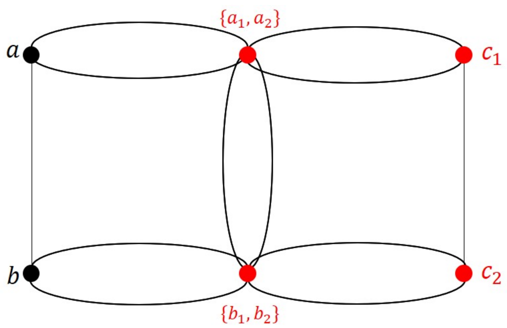


Figure 9: Initial circuit

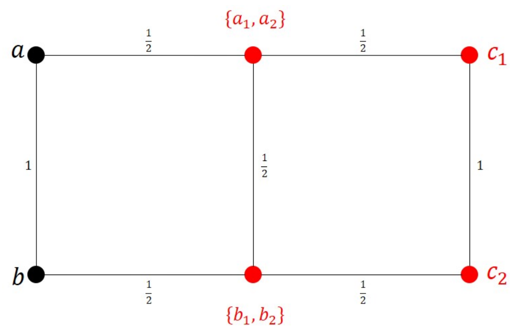
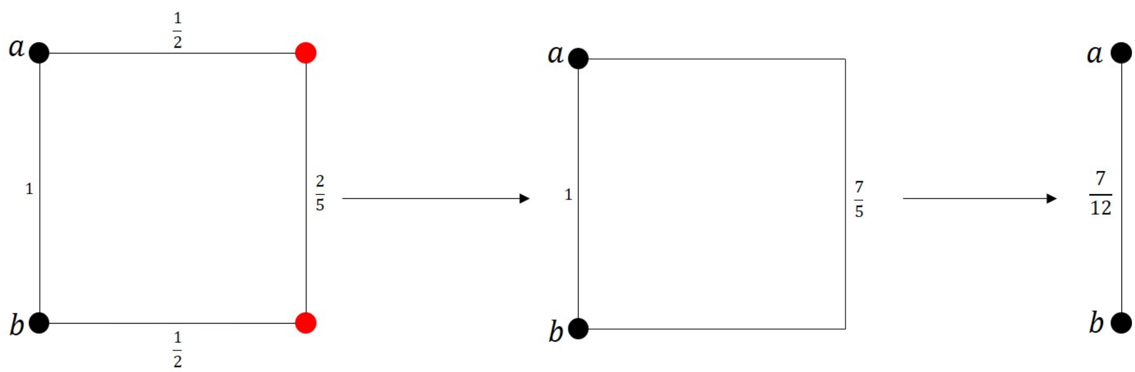


Figure 10: Intermediate circuit



Therefore from the simplification of the circuit as seen above, we conclude that the effective resistance is $R_{eff} = \frac{7}{12}$.

2. Deduce the probability that the symmetric random walk on the cube starting at vertex a reaches vertex b before returning to a.

Proof: So from the simplified circuit above in part 1), we found $R_{eff} = \frac{7}{12}$, therefore we have $i(a) = \frac{12}{7}$, so by Lemma 8.5, we conclude that the probability that the symmetric random walk starting from a reaches b before returning a is

$$p_{esc}(a,b) = \frac{c_{eff}(a,b)}{c(a)} = \frac{i(a)}{c(a)} = \frac{12}{7} \cdot \frac{1}{3} = \frac{4}{7}$$

3. Compare this escape probability with the two analogous escape probabilities obtained in Example 8.1 and Exercise 8.7.

Proof:

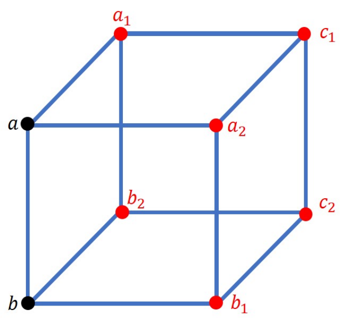


Figure 11: Exercise 8.8

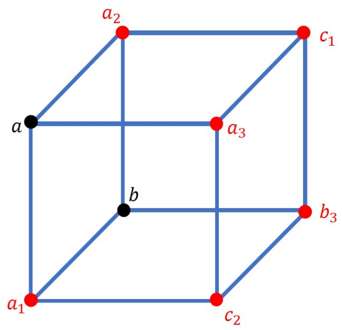


Figure 12: Exercise 8.7

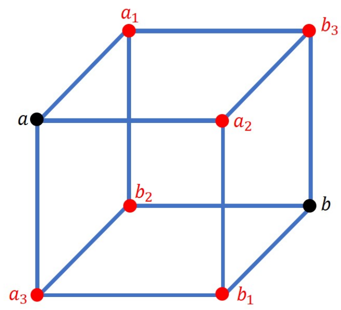


Figure 13: Example 8.1

- In Exercise 8.8 we found an escape probability of $p_{esc}(a,b) = \frac{4}{7}$.
- In Exercise 8.7 we found an escape probability of $p_{esc}(a,b) = \frac{4}{9}$.
- In Example 8.1 we found an escape probability of $p_{esc}(a,b) = \frac{4}{10}$.

Notice that the conductance at a is the same for all three examples. Therefore, the escape probability for each example scales with the conductance between a and b.

Exercise 8.9.

For the symmetric random walks on the octahedron, the icosahedron and the dodecahedron shown in Figure 7.1, compute the probability that the process starting at vertex a reaches vertex b before returning to a where vertices a and b are any two diametrically opposite vertices.

Octahedron

Proof: Consider the network below notice that the vertices a and b are two diametrically opposite vertices. Let the each edge of the network have a resistance of one unless otherwise specified.

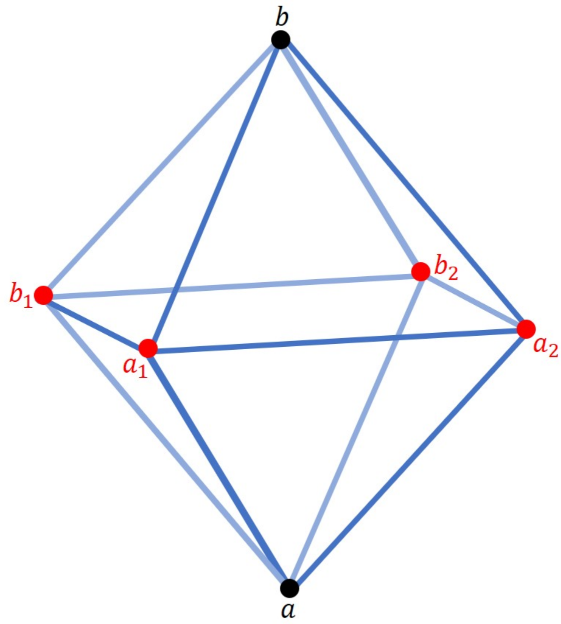


Figure 14: Octahedron.

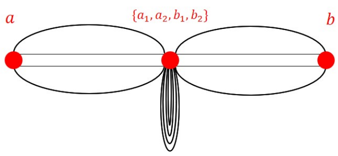


Figure 15: Initial circuit

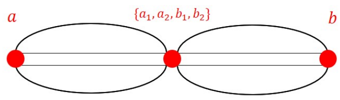


Figure 16: Intermediate circuit

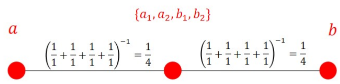


Figure 17: Simplified circuit

So from the simplified circuit in Figure 17, we find $R_{eff} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, therefore we have $i(a) = 2$, so by Lemma 8.5, we conclude that the probability that the symmetric random walk starting from a reaches b before returning a is

$$p_{esc}(a, b) = \frac{c_{eff}(a, b)}{c(a)} = \frac{i(a)}{c(a)} = 2 \cdot \frac{1}{4} = \frac{2}{4} = 0.5$$

Icosahedron

Proof: Consider the network below notice that the vertices a and b are two diametrically opposite vertices. Let the each edge of the network have a resistance of one unless otherwise specified.

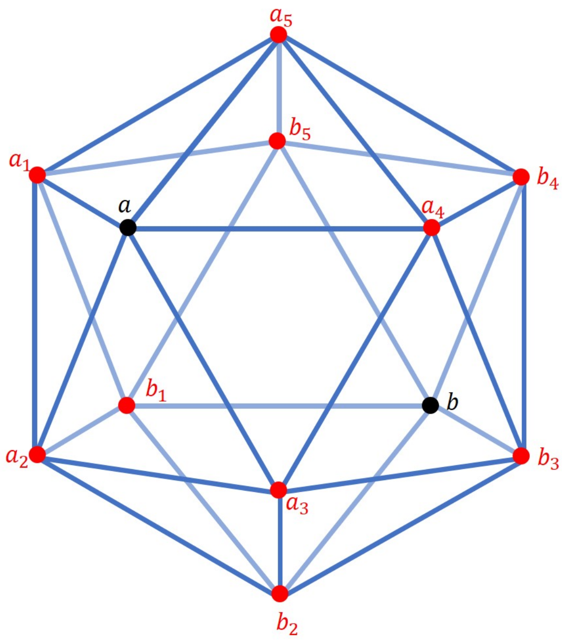


Figure 18: Icosahedron.

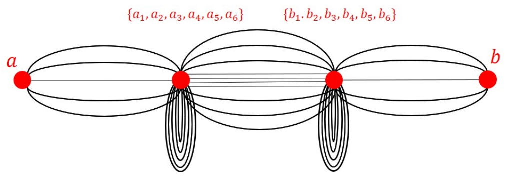


Figure 19: Initial circuit

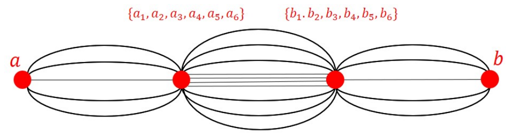


Figure 20: Intermediate circuit

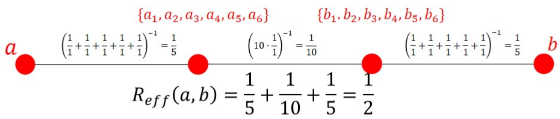


Figure 21: Simplified circuit.

So from the simplified circuit in Figure 21, we find $R_{eff} = \frac{1}{2}$, therefore we have $i(a) = 2$, so by Lemma 8.5, we conclude that the probability that the symmetric random walk starting from a reaches b before returning a is

$$p_{esc}(a, b) = \frac{c_{eff}(a, b)}{c(a)} = \frac{i(a)}{c(a)} = 2 \cdot \frac{1}{5} = \frac{2}{5} = 0.4$$

Dodecahedron

Proof: Consider the network below notice that the vertices a and b are two diametrically opposite vertices. Let the each edge of the network have a resistance of one unless otherwise specified.

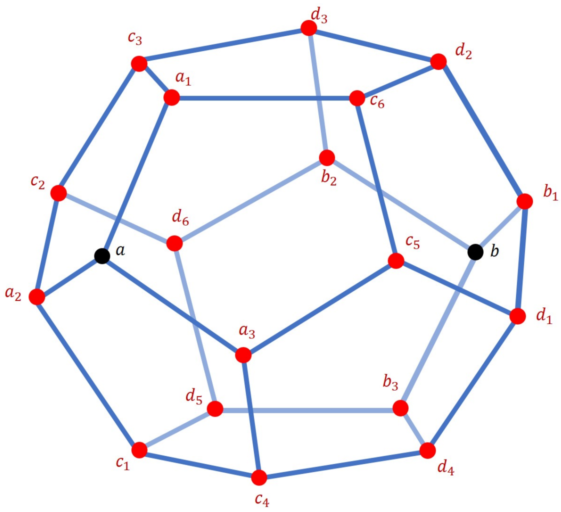


Figure 22: Dodecahedron.

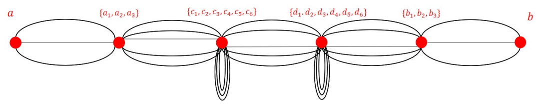


Figure 23: Initial circuit

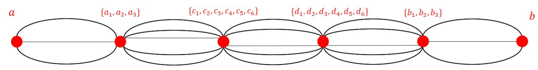


Figure 24: Intermediate circuit

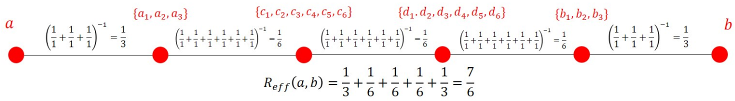


Figure 25: Simplified circuit.

So from the simplified circuit in Figure 25, we find $R_{eff} = \frac{7}{6}$, therefore we have $i(a) = \frac{6}{7}$, so by Lemma 8.5, we conclude that the probability that the symmetric random walk starting from a reaches b before returning a is

$$p_{esc}(a,b) = \frac{c_{eff}(a,b)}{c(a)} = \frac{i(a)}{c(a)} = \frac{6}{7} \cdot \frac{1}{3} = \frac{6}{21} = \frac{2}{7}$$